VISCOPLASTIC BEHAVIOUR
AND DESIGN OF TUNNELS
in loving memory of my grandmother Lina
Ambient light


tunnel atmosphere


tunnel lining


tunnel alignment


tunnel crosssection

Abstract

In the framework of Geotechnical Engineering and Rock Mechanics many tunnels are known where, even during construction, large deformations and high stresses in the lining are observed. This is often the result of squeezing behaviour. Tunnel construction in such a condition is very demanding and difficulties are met in making reliable predictions at the design stage. The selection of the most appropriate excavation-construction method to be adopted (i.e. mechanized tunnelling versus conventional tunnelling) is highly problematic and uncertain.

The present thesis is to contribute to the understanding of the squeezing behaviour of tunnels with a major interest on the time dependent response. The research is focused on the experimental investigation of time dependent characteristics of weak rock at the laboratory scale and on the formulation of a new constitutive model (SHELVIP) to be used in design practice. The considered case of study is the Saint Martin La Porte access adit, along the Torino-Lyon Base Tunnel, which experienced very important squeezing problems during excavation.

Following the introduction of the various time dependent phenomena that are known to exist for soils/rocks and that are unanimously accepted by the geotechnical community, the thesis examines the constitutive models that have been proposed so far to capture the time dependent behaviour. The bibliographic study highlights that only few models can reproduce satisfactorily all the features of the time dependent behaviour of weak rocks involved in tunnel excavation, with a simple formulation to be used in design practice. This observation led to the study of the peculiar characteristics of weak rocks and to the formulation of a novel viscoplastic constitutive model.

The availability of rock samples obtained from the Saint Martin La Porte tunnel and the peculiar characteristics of the material has determined the choice of coal as the rock material of interest for the present study. An experimental program has been performed on this material, involving the determination of the physical properties and mineralogical composition, in conjunction with oedometer tests, direct shear tests and triaxial tests, performed by using two advanced laboratory equipments: the High Pressure Triaxial Apparatus (HPTA)

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and the High Pressure Back Pressure Shear Apparatus (HPBPSA). Particular
attention has been posed on the deformability, strength characteristics, and
time dependent behaviour. The results obtained highlight that most of the
time dependent characteristics of coal follow in line the experimental evidences
reported in literature. Only few aspects differ significantly. These observations
form the background for the formulation of the new constitutive model proposed.

The SHELVIP (Stress Hardening ELastic VIscous Plastic) model, a new
viscoplastic constitutive law, has been developed to incorporate the most im-
portant features of behaviour observed in tunnels excavated in severe to very
severe squeezing conditions in a simple, but complete, manner. This model
couples the elastoplastic and time dependent behaviour by using a plastic yield
surface, as frequently adopted in tunnel design analysis, and the definition of
a state of overstress referred to a viscoplastic yield surface. The model has
been formulated in all its detailed aspects. The related analytical closed-form
solution for representing triaxial creep deformations has been developed. By
observing the behaviour of the model with reference to classical time dependent
tests, is should be noted that the SHELVIP model can reproduce almost all the
important aspects of time dependency of soils/rocks. Finally, the new model
has been implemented into the finite difference code FLAC, in order to allow
to perform numerical analyses of geotechnical problems. The SHELVIP model
has been calibrated by using the laboratory tests performed on coal specimens.
The model is shown to fit very satisfactorily the experimental results of creep
and stress relaxation triaxial tests.

In the final part of the thesis a series of numerical analyses, which have been
carried out using the newly developed SHELVIP model on a representative
section of the Saint Martin La Porte tunnel with the intent to evaluate the
ability of the model to describe the squeezing conditions with reference to a real
case study, are described. If the results of the numerical analyses are compared
with the monitoring convergence data, radial and longitudinal displacements, it
is possible to state that the agreement of the numerical results with the mean
monitored data is excellent, notwithstanding the scattering of the monitoring
data due to the heterogeneity and anisotropy of the rock mass. Also the results
of stress distribution around the tunnel perimeter are satisfactory and reliable.
It is possible to conclude that the SHELVIP model can be used with confidence
in order to reproduce by numerical analysis the behaviour of tunnels under
severe squeezing conditions.

Further developments are needed and some open questions remain to be
addressed for the study of the time dependent behaviour of weak rocks in
relation to tunnel excavation, in particular for the assessment of the stability
conditions of both the face and the heading, the timely installation of the tunnel
support, and the excavation method to be adopted.
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Chapter 1

Introduction

1.1 Framework

Tunnel construction in squeezing conditions is very demanding due to the difficulty in making reliable predictions at the design stage. During excavation such conditions are not easily anticipated, even when driving into a specific geological formation, and experience is gained on the squeezing problems encountered. Squeezing conditions may vary over short distances due to rock heterogeneity and fluctuations in the mechanical and hydraulic properties of the rock mass.

The selection of the most appropriate excavation-construction method (i.e. mechanised tunnelling versus conventional tunnelling) is highly problematic and uncertain. In mechanized tunnelling, due to the fixed geometry and the limited flexibility of the TBM (Tunnel Boring Machine), allowable space to accommodate ground deformations is restricted. On the contrary, in conventional tunnelling a considerably larger profile can be excavated initially, in order to allow for large deformations.

1.2 Problem statement

Squeezing stands for large time dependent deformations during tunnel excavation. It takes place when a particular combination of induced stresses and material properties pushes some zones around the tunnel beyond the limiting deviatoric stress at which time dependent process starts. It is closely related to the excavation and support techniques which are adopted. Deformations may terminate during construction or continue over a long period of time (Barla, 2001).

Under the most severe squeezing conditions an appropriate representation of
the tunnel response is possible only by using constitutive models which account for time dependent behaviour.

Although the time dependent behaviour of weak rocks has been widely studied in the last decades, by means of laboratory investigations and constitutive models, a complete review of the existing literature highlights that only few models can reproduce satisfactory and in a simple manner all the features of time dependency involved in tunnel excavation. This makes the design of tunnels in squeezing conditions still an open problem.

1.3 Thesis scope and objectives

The scope of the present thesis is to contribute to the understanding of the behaviour of tunnels under severe to very severe squeezing conditions by means of experimental investigations and the formulation of a new time dependent constitutive model. The considered case study is the Saint Martin La Porte access adit, of Torino-Lyon Base Tunnel, which experienced very important squeezing deformations during excavation.

The problem of time dependency is first analysed from an experimental point of view, by means of laboratory tests on intact samples of rock in closely controlled conditions. The experimental evidences are fundamental for the evaluation of the most significant factors influencing the behaviour of the material.

A new time dependent constitutive model (SHELVIP) is then developed with the intent to describe the most important aspects of squeezing influencing the tunnel response. The goal is to propose a simple constitutive law that can be used with confidence both for research and design practice.

With the intent to highlight the potentials of the newly developed model, the case of the Saint Martin La Porte tunnel, which has been excavated through the weak Carboniferous Formation, is considered. The purpose is to reproduce, by means of back analyses, the behaviour of the tunnel both in terms of convergences and radial displacements, as excavation takes place.

The following main tasks have been undertaken:

- Detailed bibliographic study, in order to identify as far as possible the experimental observations and the theories adopted in literature to describe the time dependent behaviour of soils and weak rocks.

- Adjustment of the High Pressure Triaxial Apparatus (HPTA, DIPLAB, Department of Structural and Geotechnical Engineering, Politecnico di Torino), for testing the material from the Saint Martin La Porte tunnel.
1.4 Organisation of thesis

This thesis is divided into eight chapters and two appendices. The present chapter is intended to provide a general introduction to the subject. The following two chapters are dedicated to the bibliographic study. Chapter 2 presents a complete review of the experimental observations found in literature on time dependent behaviour of geomaterials. Chapter 3 presents the constitutive models which have been developed by different authors during the last sixty years with the main purpose to describe the viscoplastic behaviour.

The next two chapters describe the experimental testing program. Chapter 4 is devoted to the description of the triaxial testing equipment. Chapter 5 describes the mechanical characterisation of coal from Saint Martin La Porte tunnel.

The last two chapters are dedicated to constitutive modelling and numerical analyses. Chapter 6 describes the novel viscoplastic constitutive model, its main features, the implementation into a numerical code, and the calibration of the constitutive parameters based on tests on coal. Chapter 7 presents the numerical back analyses performed on the Saint Martin La Porte tunnel using the newly developed constitutive model.

Finally some conclusions and suggestions for further work are made in Chapter 8.

Appendix A gives a list of the symbols and Appendix B describes the analytical derivation of the constitutive model.
Chapter 2

Experimental time dependent behaviour of soils

2.1 Introduction

Time dependent behaviour of geomaterials has been investigated extensively in the last decades, through one-dimensional and triaxial laboratory test conditions. Most of the studies found in literature have been focused on the evaluation of the behaviour of clayey soils and sands, whereas the reported studies on weak rocks are few and somewhat inconsistent. For this reason in this thesis the time dependent behaviour of weak rocks is assumed to be similar to that of cohesive soils, like clays (Cristescu and Hunsche, 1998).

The main purpose of this chapter is to present an up-to-date review of the various time and rate dependent phenomena that are known to exist for clayey soils and that are unanimously accepted by the geotechnical community. This review forms the background for the experimental study on weak rocks and for the development of a new constitutive model, which is proposed in the following.

2.2 Assumptions

Time dependence is a general term that includes a wide range of aspects, since most of the physical and chemical phenomena are not instantaneous but necessitate a long time to develop completely. In this thesis only the deferred behaviour, which is connected to the rheological properties of the solid skeleton of a soil, is considered. The deferred effects due to the variation of the interstitial pressure (consolidation, swelling) or to the alteration of the chemo-physical properties (structuration, destructuration, chemical consolidation) are neglected.
Because the time dependent behaviour of soil is rather complex and involves a wide number of factors, the following assumptions are introduced:

- Only a phenomenological point of view is considered.
- Only time dependent behaviour of intact rock is considered. Time dependent behaviour of rock joints, which plays an important role on the deferred deformation of rock masses, is neglected.
- Temperature dependence is not considered, even if its effect may be quite significant. This assumption is supported by observing that in civil engineering applications in most cases temperature effects are not an issue. For further studies on temperature effects, see Leroueil and Marques (1996).
- Only observations obtained from oedometer and triaxial laboratory tests are considered. In situ tests are not taken into account; see Al Husein (2001) for these cases.
- Dynamic effects, which are governed by the inertial forces and by the viscoplastic properties of the material, are not taken into account. Only static or quasi-static conditions are considered.

The time dependent problems associated with laboratory testing are discussed by considering the following aspects:

- Creep: development of strains within time as the effective state of stress is maintained constant;
- Relaxation: decrease of stresses while the total strains are kept constant;
- Rate dependency. The mechanical properties of the soil depend on the rate of imposed deformation;
- Accumulated effects.

The first three aspects of behaviour are associated with the principal types of tests, which are used to investigate the time dependent response of soil/rock:

- Creep tests;
- Stress relaxation tests;
- Constant Strain Rate tests (CSR tests)

both in oedometric and triaxial conditions.
2.3 Creep

Creep test is the simplest test to perform to evaluate the time dependent behaviour. The test consists in maintaining the effective state of stress constant and in measuring the deformations of the sample over time.

A creep test (strain path A→B) is illustrated in Figure 2.1. Consider a soil loaded to point A (Figure 2.1.a). At this point, a creep process is initiated by letting the effective stresses constant over time (Figure 2.1.b). As time elapses, the stress-strain state moves to B. During this process, the strain is gradually increasing. It can be concluded that during a creep test, which is characterized by a constant stress, the strain increases.

![Figure 2.1: Creep test performed at a low stress level (A → B): (a) stress-strain relationship; (b) stress history and (c) strain history](image)

2.3.1 Considerations

In the following some considerations concerning creep tests are illustrated.

**Creep at constant load or constant stress**

The requirement of constant stress during a creep test is not fulfilled in general. There are several occasions in literature where it is not clear whether reference is made to a creep test, where the stress is kept constant or the load is kept constant. There are clear differences, as illustrated in Figure 2.2.

Creep at constant stress corresponds to a point in the stress plane (point A in Figure 2.2). On the other hand, creep at constant load represents creep under stress decreasing with time (stress path A→B in Figure 2.2). This is due to the fact that the sample area increases, thereby resulting in a continuous decrease in creep stress. Only the creep at constant stress can be taken as a true creep, because the effective state of stress is maintained constant.
Drained and undrained creep

With reference to triaxial tests performed using interstitial pressure, two different definitions of creep are found in literature. In “drained creep”, the effective stresses are kept constant. In “undrained creep” the variation of the pore pressure causes the change of the effective state of stress. According to the previous definition of creep (development of strain over time at constant effective stress), it can be concluded that undrained creep does not represent a pure creep process, whereas drained creep does. However both processes will be denoted as drained and undrained creep.

Problem of reference time

The problem of reference time is a contentious issue, that affects all drained tests on clayey soils. The concern is deciding when it is possible to assume that creep strain starts (time $t_0$), because in the phase of primary consolidation coexist both the strain due to the dissipation of the excess pore pressure and the strain due to the rheological characteristics of the solid skeleton.

Much attention has been paid to the evaluation of reference time in literature, because it is crucial for estimating creep settlements in low permeability soil such as clay. Two opposite approaches were proposed:

- The reference time $t_0$ is taken as the time at the end of the primary consolidation (Mesri and Choi, 1985). This implies that the value of $t_0$ should vary with the drainage length or the thickness of soil. In other words it is like to assume that the creep strains are negligible during pore pressure dissipation.

- The reference time $t_0$ is taken as an intrinsic parameter for a given soil
2.3. Creep

(Leroueil et al., 1985; Bjerrum, 1967; Yin, 1999). It means that \( t_0 \) is independent of drainage conditions and soil thickness. In other words this hypothesis is equivalent to assume that creep occurs during the primary consolidation process.

2.3.2 Definition of creep stages

The results of creep tests performed at different stress levels in a triaxial apparatus may be plotted in a strain-time diagram with arithmetic axes, as shown in Figure 2.3.

![Figure 2.3: Definition of creep stages for triaxial creep tests at different stress levels](image)

The process can be divided into three parts:

- “Primary creep” or “transient creep”: the strain rate decreases with time;
- “Secondary creep” or “stationary creep”: the strain rate is almost constant with time;
- “Tertiary creep” or “acceleration creep”: the strain rate increases with time and takes the sample to the so-called “creep failure”.

Although the designation of a part of the creep behaviour as steady state (secondary creep) may be convenient for some analysis purpose, a true steady-state can exist only for conditions of constant structure and stress. Such a set of conditions is likely only for a fully destructured soil, and a fully destructured state is likely to persist only during deformations at a constant rate; i.e., at failure (Mitchell, 1993).
Tertiary creep may be the cause of instabilities and local collapses occurred in some tunnels, even after long periods of inactivity of the operations of excavation.

By increasing the applied stress level it is possible to observe:

- absence of creep deformation, if the stress level is below a value defined “creep threshold”;
- only the primary phase of creep. The curve A in Figure 2.3 shows this behaviour, that is frequently described as viscoelastic, i.e., complete unloading at this stage would lead to a complete recovery of strains in some time;
- primary and secondary creep. The curve B in Figure 2.3 shows this behaviour, that is frequently described as viscoplastic, i.e., complete unloading would produce only a partial recovery of the strains;
- primary, secondary and tertiary creep. The curve C in Figure 2.3 shows this behaviour. The non-reversibility of strains is evident.

The association between the creep phases and the viscoelastic and viscoplastic behaviours (Figure 2.4) is obviously only a simple idealization. However it is reasonable to think that at low stress level the behaviour is roughly viscoelastic, while at high stress level it is viscoplastic.

![Figure 2.4: Definition of creep stages and associated behaviours](image)

The stress thresholds, that define the transition between the different behaviours of the material, are difficult to define and are practically unknown. Notwithstanding some authors studied this stress thresholds in relationship to
the failure stresses, it is difficult to summarize a general criterion holding true for all soils.

**Observations on oedometer creep tests**

Some confusion exists on primary, secondary and tertiary creep phases defined in connection with creep tests performed in a triaxial apparatus (Figure 2.5), and primary, secondary, and tertiary compression phases defined in connection with step load tests performed in an oedometer apparatus (Figure 2.6).

**Figure 2.5:** Definition of creep stages for creep tests performed in a triaxial apparatus: (a) strain versus time and (b) logarithm of strain rate versus logarithm of time

**Figure 2.6:** Definition of primary, secondary, tertiary compression phases in oedometer conditions: (a) strain versus logarithm of time and (b) logarithm of strain rate versus logarithm of time

For oedometer tests, primary, secondary and tertiary compression phases
can be defined by plotting strains versus the logarithm of time. The primary compression phase is identical to the primary consolidation, where excess pore pressure dissipates. The secondary compression phase is also denoted as secondary consolidation and corresponds to pure creep, i.e., deformations occur due to deformations in the soil skeleton. Tertiary compression corresponds to pure creep, too. The tertiary compression phase is subsequent to the secondary compression phase and it is characterized by a non-linear relationship between strain and the logarithm of time.

By comparing Figures 2.5 and 2.6 it can be concluded that there are clear differences between creep and compression phases. By inspecting Figure 2.6.a, nothing can be stated about the changes in strain rate with time \( \frac{d^2 \varepsilon}{dt^2} \), because strain is plotted against logarithm of time. It can be shown from elementary definitions of \( \ln(t) \) and differentiation that:

\[
\frac{d^2 \varepsilon}{dt^2} = \frac{1}{t^2} \left( \frac{d^2 \varepsilon}{d(\ln t)^2} - \frac{d\varepsilon}{d(\ln t)} \right)
\] (2.1)

where \( t \) is the time and \( \varepsilon \) the strain. From Equation (2.1), it can be concluded that the strain rate increases if the second derivative of strain with respect to the logarithm of time is larger than the first derivative. The rate remains constant when both derivatives are equal. The normal case will be that the rate continues to decrease, for which the second derivative is smaller than the first derivative. Thus, a steepening strain versus logarithmic of time curve (tertiary compression) corresponds to a decreasing rate. In Figure 2.6.b, the logarithm of strain rate is plotted versus the logarithm of time for a single load increment in an oedometer test. It can be noted that the strain rate always decreases with time. Therefore, it can be concluded that in oedometer tests, only primary creep can be observed whereas secondary and tertiary creep cannot.

### 2.3.3 Strain-time behaviour

**Observations from oedometer tests**

Historically, the first observations on the evolution of the creep deformations with time have been done with reference to oedometer tests (problem of secondary compression).

Secondary compression is often depicted as an approximately linear relationship between the vertical strain \( \varepsilon_z \) or void ratio \( e \) and the logarithm of time \( t \). This relation is given by the coefficient of secondary compression \( C_\alpha \), as shown in Figure 2.7. This coefficient can be defined in different ways, with the most commonly used definition given by:
\[ C_{ae} = \frac{\Delta e}{\Delta \log(t)} \; ; \quad C_{a\varepsilon} = \frac{\Delta \varepsilon_z}{\Delta \log(t)} = \frac{\Delta e}{(1 + e_i) \Delta \log(t)} = \frac{C_{ae}}{1 + e_i} \] (2.2)

in which \( e_i \) is the initial void ratio and \( C_{ae} \) and \( C_{a\varepsilon} \) are the secondary compression coefficients, with respect to void index \( e \) and vertical strain \( \varepsilon_z \) respectively.

**Figure 2.7:** Definition of the coefficient of secondary compression \( C_{ae} \) for an oedometer test

If Equation (2.2) is rewritten with respect to \( \varepsilon_z \), the logarithmic relation which is used to model the secondary compression is:

\[ \varepsilon_z = C_{a\varepsilon} \log \left( 1 + \frac{t}{t_0} \right) \] (2.3)

where \( t_0 \) is some reference time.

One of the major difficulties when using Equation (2.3) is deciding when the creep deformation starts, or, in other words, determining the reference time \( t_0 \). This is the problem of the reference time discussed in Section 2.3.1.

The linear relationship between creep strain and logarithm of time may be valid for several oedometer tests, but it does not hold true in general. A non-linear behaviour was observed by Bjerrum (1967), Yin (1999) and Leroueil et al. (1985). The curves for over consolidated specimens, type A in Figure 2.8, show a continuously increasing slope with the logarithm of time after the end of primary consolidation (E.O.P.). On the other hand, normally consolidated specimens, type C in Figure 2.8, show a continuously decreasing slope.

In order to better understand the behaviour observed during oedometer tests, the logarithm of vertical strain rate can be plotted versus logarithm of
Chapter 2. Experimental time dependent behaviour of soils

Figure 2.8: Different types of strain-time curves in an oedometer test. Type A corresponds to an over consolidated sample. Type B corresponds to a sample where the stress is close to preconsolidation pressure. Type C is a normally consolidated sample. (EOP=End Of Primary consolidation)

time, as shown in Figure 2.9.a. In this diagram a straight line is characterized by the $m$ parameter according to Singh and Mitchell (1968):

$$m = \frac{\Delta \log(\dot{\varepsilon})}{\Delta \log(t)}$$

where, in one-dimensional tests, $\dot{\varepsilon}$ can be chosen as the vertical strain rate $\dot{\varepsilon}_z$.

As shown in Figure 2.9, the straight line which represents the Terzaghi primary consolidation process has a slope $m = 0.5$. For an over consolidated soil it is approximately $m = 0.8$. For a normally consolidated soil $m = 1.0$, excluding the first part of the curve where the slope is similar to one of Terzaghi primary consolidation.

The S-shape of the soil tested at a stress level close to preconsolidation pressure was observed by Tavenas et al. (1978) and corresponds to a delayed creep, caused by the excess pore pressure during consolidation. This transition corresponds to a transition from over consolidated to normally consolidated creep states (Kabbaj et al., 1986)

As mentioned above, a third oedometer phase, called tertiary compression, has been observed after secondary compression. This phase is not well documented in literature.
2.3. Creep

FIGURE 2.9: (a) Different types of strain-time curves in an odometer test in the \( \log(\dot{\varepsilon}_z) - \log(t) \) diagram and (b) illustration of the characteristic \( m \) values in \( \log(\dot{\varepsilon}_z) - \log(t) \) diagram. Type A corresponds to an over consolidated sample. Type B corresponds to a sample where the stress is close to preconsolidation pressure. Type C is a normally consolidated sample.

Observations from triaxial tests

In literature there are relatively few reports of drained creep tests on clays, compared with the number of undrained tests. In the following, emphasis is placed on the observations from drained conditions (true creep conditions) whenever it is possible.

In order to visualize the experimental behaviour in triaxial conditions, the test data are again plotted in \( \log(\dot{\varepsilon}) - \log(t) \) diagrams. When there is not tertiary creep, i.e., the stress state is far from failure, the experimental creep curve can be represented with a sufficient approximation with a straight line defined by the \( m \) Sigh and Mitchell’s parameter of Equation (2.4). However, at first sight, it may be quite difficult to imagine the consequences of varying the \( m \) value. For that reason, the characteristics of three different \( m \) values are illustrated in Figure 2.10.

Figure 2.11 shows triaxial creep tests carried out on undisturbed Saint-Alban clay after Tavenas et al. (1978). Excluding the tertiary phases of creep, it is possible to observe a linear trend.

Some authors confirm the hypothesis of linear trend: Tavenas et al. (1978), Singh and Mitchell (1968) and Bishop and Lovenbury (1969) for OC clays, while other authors do not: Bishop and Lovenbury (1969) for NC clays. However, it is possible to conclude that this hypothesis is a quite good approximation, which is rather useful to model the time dependent behaviour of clayey soils.
Chapter 2. Experimental time dependent behaviour of soils

**Figure 2.10:** Creep characteristics for three different $m$ values. $m = 1$ corresponds to a straight line in the $\varepsilon_a - \log(t)$ diagram. $m \neq 1$ corresponds to curved lines in the $\varepsilon_a - \log(t)$ diagram.

**Figure 2.11:** Oedometer and triaxial drained creep tests carried out on undisturbed Saint-Alban clay after Tavenas et al. (1978).
Tavenas et al. (1978) noticed that the same parameter $m$ can be used to describe both the volumetric and deviatoric creep strains. In general this is not likelihood: even thought the volumetric strains can be represented with a straight line in the $\log(\dot{\varepsilon}) - \log(t)$ diagram, as well as the deviatoric strains, the slope is not equal to the one of the deviatoric strains (Feda, 1992; Tian et al., 1994).

The values of $m$ in triaxial creep tests reported by various authors, highlight that, within the normally consolidated range, $m$ varies between 0.7 and 1.2 with the most values less than 1.0; in the over consolidated range the $m$ values seem to be less than 1.0.

As depicted in Figure 2.11, when tertiary creep develops the trend goes out from the straight line and the strain rate increase very rapidly, leading the sample to failure. The duration and the progression of the tertiary phase of creep is not well known, therefore it is impossible to define a general behaviour during this phase.

### 2.3.4 Stress dependency

The effects of the effective stress on the development of deferred strains during a creep test have been the subject of numerous investigations both for oedometer and triaxial test conditions.

**Observations from oedometer tests**

The first observation on the stress dependency in oedometer tests leads to a secondary compression coefficient $C_{\alpha\varepsilon}$ linearly related to the compression index $C_{ce} \ (C_{ce} = \Delta\varepsilon / \Delta \log \sigma'_z)$ over the entire applied stress range. The average ratio between the secondary compression coefficient and the compression index $C_{\alpha\varepsilon} / C_{ce}$ was found to be in the range 0.025 to 0.10, with the higher values applying to the high organic plastic clays (Mesri and Godlewski, 1977). It was concluded that $C_{\alpha\varepsilon}$ and $C_{ce}$ depend on the applied effective stress $\sigma'_z$ and on the preconsolidation pressure $\sigma'_{z,pc}$. It was shown that both $C_{\alpha\varepsilon}$ and $C_{ce}$ increase as the effective stress $\sigma'_z$ approaches the preconsolidation pressure $\sigma'_{z,pc}$, then decrease and finally remain reasonably constant. Throughout these effective stress changes the ratio $C_{\alpha\varepsilon} / C_{ce}$ remains reasonably constant.

Several authors questioned the uniqueness of the $C_{\alpha\varepsilon} / C_{ce}$ concept in which $C_{\alpha\varepsilon}$ and $C_{ce}$ are assumed to be time independent. They concluded that not only $C_{\alpha\varepsilon}$ but also $C_{ce}$ changes with time. Furthermore they reported that the changes in $C_{\alpha\varepsilon}$ with time might reflect the changes in $C_{ce}$ with time, which means that the relationship between $C_{\alpha\varepsilon}$ and $C_{ce}$ holds true for any time, effective stress, and deformation.
Observations from triaxial tests

Most of the experimental evidences found in literature shows that the creep strain depends only on the deviatoric component of stress applied to the material (Mitchell, 1993).

The influence of the stress magnitude on the creep strain rate, at a given time $t$, is shown in Figure 2.12. At low stress the creep rate is small and of little practical importance. In the midrange of stress, a nearly linear relationship is found between the logarithm of strain rate and stress. With stress approaching the strength of the material, the strain rate becomes very large and signals the onset of failure. An example of the relationship between the logarithm of strain rate and stress is given in Figure 2.13.a with reference to undrained triaxial creep tests on remoulded illite (Mitchell, 1993). Results from drained tests on London clay (Bishop, 1966) are shown in Figure 2.13.b; only values in the midrange of stress are shown in this figure.

![Figure 2.12: Influence of stress magnitude on creep strain rate at given time $t$](image)

Singh and Mitchell (1968) suggest that for a given soil the value of the parameter $m$ is independent of the deviatoric stress level. In other words, the creep curves for different stress levels have the same slope in the $\log(\dot{\varepsilon}) - \log(t)$ diagram.

Some authors found that the parameter $m$ does not change with the applied deviatoric stress (Bishop and Lovenbury, 1969; Feda, 1992; Tian et al., 1994; Zhu et al., 1999). Other authors instead found that $m$ increases with the deviatoric stress (Bishop and Lovenbury, 1969; Feda, 1992; Tian et al., 1994), or that the opposite is true (Zhu et al., 1999).

In all cases, given the difference between the various experimentations, it is possible to state that the hypothesis of $m$ independent from the deviatoric
2.3. Creep

Figure 2.13: Variation of creep strain rate with deviatoric stress: (a) undrained creep tests on remoulded illite (Mitchell, 1993) and (b) drained creep tests on London clay (Bishop, 1966)

stress is a quite good approximation.

2.3.5 Creep failure

Some soils may fail under a sustained creep stress significantly less (as much as 50%) than the peak stress measured in a standard compression test, where a sample is loaded to failure in a few minutes or hours. This is termed “creep rupture” or “tertiary phase of creep”.

The loss of strength as a result of creep may be explained considering that a significant portion of the strength of a soil is due to cementation and creep deformations cause failure of cemented bonds.

The critical stress below which creep rupture does not occur is termed “upper yield” Mitchell (1993). This limit has been subjected to few experimental
investigations and it is practically unknown.

**Time to failure**

For soils subjected to creep failure the time to failure $t_f$ is usually a negative exponential function of the deviatoric applied stress $q$. The relationship between deviatoric stress and time to failure for Haney clay is shown in Figure 2.14 (Campanella and Vaid, 1972).

![Image](image.png)

**Figure 2.14:** Time to rupture as a function of creep stress for Haney clay (Campanella and Vaid, 1972)

The minimum creep strain rate $\dot{\varepsilon}_{\text{min}}$ prior to the onset of creep rupture increases and the time to failure decreases, as the stress intensity increases, as shown in Figure 2.15 for Haney clay. The relationship is unique, as may be seen in Figure 2.16 which shows that:

$$t_f = \frac{C}{\dot{\varepsilon}_{\text{min}}}$$  \hspace{1cm} (2.5)

where $C$ is constant for a given soil.

The strain at failure is a constant independent of stress level, as shown in Figure 2.17. The failure strain is taken as the strain corresponding to the minimum strain rate $\dot{\varepsilon}_{\text{min}}$. 

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2.3. Creep

Figure 2.15: Creep rate behaviour for $K_0$-consolidated undisturbed Haney clay (Campanella and Vaid, 1972)

Figure 2.16: Relationship between time to failure and minimum creep rate (Campanella and Vaid, 1974)
Chapter 2. Experimental time dependent behaviour of soils

2.3.6 Representation of results

The rheological behaviour of a material can be investigated by performing a series of creep tests, each of them at a different deviatoric stress level and keeping all the other parameters constant. The experimental results can be represented with an interesting diagram proposed by Ladanyi (1993) and shown in Figure 2.18.

In the quarter A of Figure 2.18 the creep strain versus time are reported. If the points, which represent the starting times of tertiary creep, are plotted as a function of the applied deviatoric stress, the “delayed strength” curve is obtained in the quarter B. This curve starts with the “short-term strength” $\sigma_{st}$ and tends asymptotically to the “long-term strength” $\sigma_{lt}$, also called “creep limit” or “creep threshold”.

If the minimum creep rates are plotted as a function of the applied stress, it is possible to obtain a curve in quarter C. This curve is called “rheological curve” of the material and characterises the rate sensitivity of the strength. This curve can also be obtained by performing a series of compression tests at various controlled strain rates and by plotting the peak stresses against the applied strain rates. For a material with a real creep limit, this curve does not pass through the origin but starts at the creep limit stress level $\sigma_{lt}$.

Finally, quarter D represents a set of isochronous stress-strain curves, obtained by intersecting the creep curves in quarter A at various constant times, and plotting stresses against the strains at intersection points. These curves cannot be obtained by direct testing.
Figure 2.18: Schematic representation of results from uniaxial compression creep tests (Ladanyi, 1993)
2.4 Stress relaxation

Stress relaxation is a rather simple test, which is frequently used to evaluate the time dependent behaviour of soils, and represents the dual aspect of creep test. This test consists in maintaining the displacement of the sample constant and in measuring the variation of the effective state of stress with time. It can be performed both in oedometer and triaxial conditions.

As this kind of test necessitates much less time than a creep test (it is important to remember that a creep test could last a long time, as months or years), it is frequently preferred in order to quickly evaluate the time dependent behaviour of soils.

A stress relaxation test (stress path $A \to B$) is illustrated in Figure 2.19. Consider a soil loaded to point $A$. At this point, a stress relaxation process is initiated by letting the strains constant over time. As time goes by, the stress-strain state moves $B$. During this process, the stress is gradually decreasing. It relaxes. Therefore, it can be concluded that during a stress relaxation test, which is characterized by constant total strain, the stress decreases.

![Figure 2.19: Stress relaxation test (A → B): (a) stress-strain relationship; (b) strain history and (c) stress history](image)

2.4.1 Considerations

Most of the triaxial relaxation tests, which have been presented in the literature, have been performed in undrained conditions, by maintaining constant the vertical displacement of the sample and the confinement pressure. This test cannot be considered as a true stress relaxation test, because, strictly speaking, a true relaxation test requires drained condition and keeping constant both the vertical displacement and the radial displacement of the sample (complex procedure).
2.4. Stress relaxation

2.4.2 Stress-time behaviour

There are very few reported investigations that treat stress relaxation of clay in oedometer test conditions. The experimental studies on triaxial conditions are anyhow few but it is possible to highlight a “reference relaxation behaviour” which has been presented by Lacerda and Houston (1973) and which acts as a basis for other investigations on relaxation.

Lacerda and Houston (1973) reviewed the few previous investigations, performed several stress relaxations tests (on N.C. San Francisco Bay Mud, kaolinite, Monterey sand and Ygnacio Valley Clay), and made contributions to the modelling aspect of time dependent behaviour.

For these authors the decay of the deviatoric stress $q$, normalized to the deviatoric stress at the beginning of stress relaxation $q_0$, is linear with the logarithm of time $t$ at first approximation, with the exception of the first phase where the trend is strongly non linear. Figure 2.20 shows this behaviour.

![Figure 2.20](image)

**Figure 2.20:** Stress relaxation test: (a) stress-strain diagram showing three different relaxation tests where the strain rate prior to the relaxation varies and (b) normalized stress versus logarithm of time for the three relaxation tests

The value of the applied strain rate during the compression phase preceding the stress relaxation phase influences the time at which the stress relaxation begins. Lacerda and Houston (1973) observed that when slow strain rate were used, there was a time delay prior to the initiation of deviatoric stress decay. This time delay seems to be inversely proportional to the strain rate prior to the relaxation phase. This behaviour is illustrated in Figure 2.20.b.

Vialov and Skibitsky (1961), Sheahan et al. (1994), Zhu et al. (1999) and Silvestri et al. (1988) reported the existence of a final relaxed stress, which was not found by Lacerda and Houston (1973). They suggest the existence of a “static stress surface” connecting the points which represent the relaxed stress states.
Generally during undrained stress relaxation tests only small or none pore pressure development was observed (Silvestri et al., 1988; Sheahan et al., 1994; Zhu et al., 1999)

### 2.5 Rate dependency

Rate dependency has been investigated extensively in the past and several authors have confirmed that rate effects have great influence on the stress-strain behaviour of clay.

#### 2.5.1 Constant strain rate

In Constant Strain Rate (CSR) tests the strain rate \( \dot{\varepsilon} = \frac{d\varepsilon}{dt} \) is enforced and kept constant throughout the experiment. The stress response is then measured in order to obtain a stress-strain relationship.

This type of test can be carried out both in one-dimensional and triaxial conditions. In oedometer tests various continuous loading testing procedures were introduced, in order to reduce the time involved in performing a consolidation test and to obtain a continuous stress-strain curve.

**Observations from oedometer tests**

Oedometer tests performed by Leroueil et al. (1985) on different natural clays, suggest that the strain rate can influence the preconsolidation pressure and the stress-strain behaviour.

The general observation is that the faster the loading rate, the higher the effective stress for a certain strain. Figure 2.21 shows this behaviour. It is possible to observe that the compression curves move to the right for higher strain rates.

Moreover, the increase of the loading rate produces the increase of the obtained preconsolidation pressure. Figure 2.22 gives the variation of the preconsolidation pressure with the strain rate for oedometer tests performed on Batiscan clay by Leroueil et al. (1985).

Leroueil et al. (1985) suggest that the behaviour is controlled by a unique relationship between effective stress, strain and strain rate \( (\sigma_z' - \varepsilon_z - \dot{\varepsilon}_z) \). This unique relation is denoted “isotach behaviour”. Leroueil et al. (1985) showed that this unique relation could be described by two curves, one giving the effective stress-strain relationship normalized to the preconsolidation pressure and the other one giving the variation of the preconsolidation pressure with the strain rate.
2.5. Rate dependency

Figure 2.21: Oedometer test performed at different strain rates on Batiscan clay (Leroueil et al., 1985)

Figure 2.22: Variation of the preconsolidation pressure with strain rate for oedometer test performed on Batiscan clay (Leroueil et al., 1985)
Observations from triaxial tests

Experimental results obtained from constant strain tests performed in triaxial conditions show the increase of the slope of the stress-strain loading curve and the increase of the peak strength with the increase of the applied strain rate. Figure 2.23 depicts these effects for over consolidated Saint-Jean-Vianny clay. The tests were performed in undrained conditions by Vaid et al. (1979).

Figure 2.23: Stress-strain behaviour of Saint-Jean-Vianny clay in undrained constant strain rate tests (Vaid et al., 1979)

Tavenas and Leroueil (1977) indicated that the effects of loading rate on the preconsolidation pressure during oedometer tests can be generalized to the entire limit state surface of the soil. Similar results were found by Tavenas et al. (1978) for over consolidated Saint-Alban clay and by Zhu et al. (1999) for soft Honk Kong marine deposit.

The strength envelope for normally consolidated soil (i.e. the critical state line) seems to be independent of rate effects.

2.5.2 Change of rate of strain

The existence of a unique relationship between effective stress, strain and strain rate has been confirmed by special constant strain rate oedometer tests, in which the strain rates were changed at various strains (Leroueil et al., 1985). The results, which are depicted in Figure 2.24, clearly show the unique stress-strain-strain rate relationship. An important feature is that the effects of change
in rate are continuous, that is the soil “stays” on the same stress-strain curve until the strain rate is changed again.

Figure 2.24: Special constant strain rate oedometer tests on Batiscan clay (Leroueil et al., 1985)

2.6 Accumulated effects

The accumulated can be observed in the stress-strain relationship subsequent to long period of ageing, due to drained or undrained creep. It is possible to consider three different types of post-ageing stress-strain relationships (Tatsuoka et al., 2000), as shown in Figure 2.25:

- Type 1, ageing without structuration: the stress-strain relationship rejoins the original loading curve without exhibiting overshooting.

- Type 2, temporary structuration effects: the stress-strain relationship rejoins the original loading curve after having exhibited a temporary overshooting.
Chapter 2. Experimental time dependent behaviour of soils

- Type 3, persistent structuration effects: the stress-strain relationship does not rejoin the original loading curve. There is a persistent overshooting with noticeable larger peak strength.

![Diagram showing different types of post-ageing stress-strain relationship](image)

**Figure 2.25:** Different types of post-ageing stress-strain relationship

Because of the hypotheses of this review only the first type of ageing can be considered as a true effect of time dependence. The other two types of ageing are due to a chemo-physical structuration, which is not possible to explain by using the isotach behaviour. For this reason only the first type of ageing, ageing without structuration, is discussed in the following section.

### 2.6.1 Ageing without structuration

The ageing without structuration can be observed both in oedometer conditions and triaxial conditions.

**Observations from oedometer tests**

When a soil is loaded to a constant stress for a long period of time, the void ratio decrease progressively as depicted in Figure 2.26. If the soil is then reloaded, an increase of stiffness can be observed and the stress-strain state moves to the curve of constant strain rate corresponding to the new strain rate. The compression curve shows a preconsolidation pressure $\sigma'_{z,pc1}$ associated to the new strain rate, which is larger than the initial one $\sigma'_{z,pc0}$. The development of this quasi-preconsolidation, or apparent preconsolidation pressure, was observed by Bjerrum (1967).
2.7 Influencing factors

Observations from triaxial tests

The effect of ageing can be extend to the entire limit state surface, similarly to what done in the previous subsection. It is important to notice that an expansion of the yield surface can be due also to the chemo-physical structuration of the soil. This topic is not considered in the present review.

2.7 Influencing factors

In this section some factors which may influence the time dependent response of soils are discussed with reference to clay.

2.7.1 Composition

Generally, the higher the clay content and the more active the clay, the more important are the time dependent effects as illustrated in Figures 2.27 and 2.28 where steady state creep rates are related to clay type, clay content and plasticity. In these tests, environmental factors were held constant by preparing all specimens to the same initial conditions (isotropic consolidation of saturated sample to 200 kPa) and application of a creep stress equal to 90% of the strength determined by a normal strength test. The correlation between the strain rate and the plasticity index is reasonably unique because the plasticity index reflects both the type and the amount of clay.
Figure 2.27: Effect of amount and type of clay on steady state creep rate (Mitchell, 1993)

Figure 2.28: Relationship between clay content, plasticity index and creep rate (Mitchell, 1993)
2.7. Influencing factors

2.7.2 Interstitial water

Time dependent deformations are more important at high water content than at low. The magnitude of creep strains and strain rate may be small in a dry soil.

Most of the reported experimental tests were performed under undrained conditions while those performed under drained conditions are few. Moreover in literature there are no experimental studies of the effects of pore pressure on the time dependent behaviour of soil. Therefore the only accepted effect of pore pressure on time dependency is to change the effective state of stress according to the Terzaghi principle.

2.7.3 Temperature

An increase in temperature increases pore pressure, decreases effective stress and weakens the soil structure. Creep rates ordinarily increase and the relaxation stress corresponding to specific values of strain decreases at higher temperature. These effects are illustrated by the data shown in Figures 2.29 and 2.30.

![Figure 2.29: Creep curves for Osaka clay tested at different temperatures; undrained triaxial compression (Murayama, 1969)](chart)

2.7.4 Test type and stress system

Most of the experimental studies on time dependent behaviour of soils have been performed in oedometer conditions or in triaxial conditions, on sample
Chapter 2. Experimental time dependent behaviour of soils

Figure 2.30: Influences of temperature on the initial and final stresses in stress relaxation tests on Osaka clay; undrained triaxial compression (Murayama, 1969)

Figure 2.31: Creep curves for isotropically and $k_0$-consolidated samples of Haney clay tested in triaxial and plane strain compression (Campanella and Vaid, 1974)
2.8. A comprehensive approach

In this section a comprehensive approach to the different time dependent aspects of soil behaviour, is illustrated.

2.8.1 Experimental evidences

The experimental evidences that constitute the fundamental basis of this approach are explained with reference to the different tests discussed above.

Static stress surface

Silvestri et al. (1988) observed that during stress relaxation test the deviatoric stress $q$ does not diminish until zero, but reaches a final relaxed level after a given period of time. They suggested the existence of a “static surface” in the effective stress space which joins the final relaxed stress points. It is noteworthy that this static stress surface is similar to the “static yield surface” of Perzyna’s overstress theory (Perzyna, 1966). Figure 2.32 illustrates this static stress surface, with reference to triaxial stress relaxation tests.

Static stress-strain curve

As shown in Section 2.5 with reference to constant strain rate tests, the behaviour of soils strongly depends on the applied strain rate. The slope of the stress-strain curve and the value of the peak strength usually increase with the increase of the strain rate.

consolidated isotropically. However, in nature most soils have been subjected to an anisotropic stress history and deformation conditions conform more to plane strain than triaxial. Some investigations have been made to examine these factors: the general form of the stress-strain-time and stress-strain rate-time relationships are similar but the actual values may differ considerably.

For example, undisturbed Haney clay, was tested both in triaxial compression and plane strain by Campanella and Vaid (1974). Samples were normally consolidated both isotropically and under $k_0$ conditions to the same vertical effective stress. Samples consolidated isotropically were tested in conventional triaxial compression test, while sample $k_0$-consolidated were tested both in $k_0$-triaxial and plain strain conditions. The results illustrated in Figure 2.31 show that the pre-creep stress history has a significant effect on the deformations. The plane strain and the $k_0$ triaxial tests give about the same creep behaviour, which suggests that the intermediate principal stress may not be a factor of major importance.
Recently, the advances in laboratory testing allowed one to evaluate the time dependent behaviour of soils with a loading test rate (in term of strain rate) much smaller than in the past.

Sulem (1983) indicated that it is possible to obtain a limit curve into the stress-strain diagram if the applied strain rate, \( \dot{\varepsilon} \), is infinitely slow. This limit curve is called “static stress-strain curve”. Figure 2.33 depicts this particular behaviour.

![Figure 2.32: Illustration of the static stress surface obtained from stress relaxation tests](image)

![Figure 2.33: Illustration of the static stress-strain curve, instantaneous stress-strain curve and peak strength curve](image)
2.8. A comprehensive approach

**Instantaneous stress-strain curve**

If the strain rate of a constant strain rate test is very large it is possible to obtain a curve in the stress-strain diagram which represents the instantaneous behaviour. This curve is called “instantaneous stress-strain curve”. It is schematically represented in Figure 2.33.

**Peak strength curve**

By increasing the strain rate of a constant strain rate test it is possible to observe that the peak strength increases while, the deformation at the peak decreases, as shown in Figure 2.33 (Stagg, 1968). The curve which joins all the peak strength points into the stress-strain diagram is called “peak strength curve” (Figure 2.33).

**Static yield surface**

The locus of peak strength points obtained from various “static” constant strain rate tests, performed with a strain rate very small, define a “static yield surface”. This is generally assumed to be identical to the “static stress surface” defined from stress relaxation tests as discussed in 2.8.1.

![Figure 2.34: Schematics of the static stress-strain curve, instantaneous stress-strain curve, peak strength curve and static yield surface](image)
2.8.2 Interpretation of time dependent behaviour

In this section the stress relaxation and creep behaviour are explained with reference to this comprehensive approach.

The main assumption is that the isotach behaviour always holds true. It is important to observe that, with the hypotheses discussed above, all the possible stress-strain states of a soil are bounded between the instantaneous curve and the static curve, depending on the applied strain rate.

Stress relaxation

Consider a constant strain rate test performed with a strain rate greater than zero. The obtained stress-strain curve shows stress levels greater than the corresponding static curve, as shown in Figure 2.35. If the loading phase is stopped before reaching failure (point A of Figure 2.35) and the deformations of the sample are kept constant, the sample is now in an over-stressed state with respect to the static curve (long term equilibrium state). Therefore, the stress starts to decrease until the stress state reaches the static curve. This behaviour is represented in $A \rightarrow B$ path in Figure 2.35.

Creep

Consider a constant strain rate test performed with a strain rate greater than zero. The obtained stress-strain curve shows stress levels greater than the corresponding static curve, as shown in Figure 2.36. If the test is stopped before...
reaching failure and the effective state of stress is maintained constant with time, the sample is in an over-stressed state with respect to the static curve. Therefore deferred strains start to develop until a new state of equilibrium is reached, i.e. creep develops.

In this case there are three possibilities:

- The stress state is below the maximum stress level of the static stress-strain curve (point A of Figure 2.36). The creep strains start to develop until the stress-strain point reaches the static curve. In this case only primary creep develops.

- The stress state is approximately equal to the maximum stress level of the stress-strain curve (point B of Figure 2.36). The creep strains starts to develop but the stress-strain point cannot reach in a finite period of time either the static stress-strain curve or the peak strength curve. In this case primary and secondary creep develop. Because it is not physically possible that the secondary creep (with constant strain rate) continues for infinite period of time, the achievement of a constant strain level (if the point is lightly lower than the maximum of the static curve) or the development of the tertiary phase of creep (if the point is lightly larger than the maximum of the static curve) is only a question of time.

- The stress state is greater than the maximum stress level of the static stress-strain curve (point C of Figure 2.36). Creep starts to develop and it is possible to observe primary, secondary and tertiary phases of creep. The sample fails for tertiary creep when the peak strength curve is reached. It
is important to remember that a creep test is performed by controlling
the axial load, therefore when the peak strength is reached the failure
develops in an irreversible and very rapid manner.

2.9 Conclusions

This chapter presents a review of the experimental aspects of time dependent
behaviour of soils which are commonly accepted by the geotechnical community.
This review is focused essentially on clays, because the experimental observations
on weak rocks are few and somehow inconsistent. Time dependent behaviour of
soft rocks is similar to that of over consolidated clays.

The main hypothesis of this chapter is to treat the phenomena only from a
phenomenological point of view and to consider time dependence as related to
the rheological characteristics of the solid skeleton of the soil. The description
is focused on one-dimensional and triaxial test conditions, and is separated
into reported characteristics of creep, stress relaxation, rate dependency and
accumulated effects. Finally a comprehensive approach to the time dependent
behaviour of soils is presented.
Chapter 3

Constitutive models for
time dependent behaviour of soils

3.1 Introduction

A major goal of the research on time dependent behaviour of soils is to enable the development of constitutive models for use in the solution of geotechnical problems, which require the determination of deformations, displacements, strength and stability changes with time, in order to obtain realistic solutions.

A great number of constitutive models have been proposed. Different approaches have been used to capture the various time dependent phenomena (creep, stress relaxation, rate dependence and accumulated effects): empirical models based on the fitting of experimental data, extension of rate process theories, rheological models, general stress-strain-time concept models, and advanced theories of viscoplasticity. Owing to the complexity of the time dependent phenomena, it is not surprising that a sufficiently general model that can describe all the aspects of time dependence with reference to all geotechnical problems is not yet available.

The main purpose of this chapter is to present a review which categorizes and describes the basic features of the existing models as well as their advantages and limitations. The different models are classified in an attempt to clarify the confusion which occurs in literature.

In this review time and time dependence are assumed to be related only to the rheological properties of the solid skeleton of the soils, such creep, stress relaxation, rate dependence and accumulated effects. The main assumptions presented in Section 2.2 are assumed to be valid also in the following.

The constitutive models developed for weak rocks are relatively few, comparing to the wide number of models proposed for soils. As done in the previous
chapter, the following description is focused on the models proposed to describe the experimental behaviour of clayey soils. The constitutive models proposed for sands are not treated.

In this review the constitutive models are classified as follows:

- **Empirical models.** They are obtained by fitting the experimental data from creep, stress relaxation or constant strain rate tests. The constitutive relationship are generally given by closed-form solutions or differential equations. They are limited to specific cases, boundary and loading conditions. The relationship given by these models can be used as a basis for the formulation of more advanced and complete constitutive models.

  They can be subdivided in: - Primary empirical models - Secondary semi-empirical models.

- **Rheological models.** These models are proposed with reference to uniaxial conditions and they are given by closed-form solutions or differential equations. They allow to understand, in a quite simple manner, the time dependent behaviour of soils. Frequently they constitute the conceptual starting point for the development of more complicated constitutive models.

  They can be subdivided in: - Analogical models - Engineering theories of creep.

- **General theories.** These theories are generally three-dimensional and represent the most advanced aspects of numerical modelling. The main characteristic is that they are not limited to specific cases, but can describe all possible stress paths and boundary conditions. These models can be readily implemented in various numerical analysis codes, as finite element or finite difference codes.

  They can be subdivided in: - Perzyna’s overstress theory - Non Stationary Flow Surface (NSFS) theory.

In this review emphasis is placed on the description of the overall structure of the three groups of constitutive models. Particular attention will be placed on the description of the general theories.

### 3.2 Empirical models

The empirical models are empirical constitutive relations, in analytical or in differential form, obtained by fitting the experimental data from laboratory tests, such as creep, stress relaxation and constant strain rate tests. They can
be applied only to problems of specific boundary and loading conditions (e.g. one specific for creep and another one for relaxation) and frequently they involve natural time. These models reflect the real behaviour of soils and can be used as the fundamental basis to develop more sophisticated constitutive models. They provide practical solutions to engineering problems where the boundary and loading conditions are not too far from the experimental testing conditions.

The empirical model can be categorized in two groups as follows:

- Primary empirical relations. They are represented by analytical expressions that are obtained by fitting the experimental results of laboratory tests with simple mathematical equations. They accurately describe the behaviour of soils during tests, but are strictly limited to the current phenomenon (i.e. the relationship for creep cannot describe stress relaxation and vice versa).

- Secondary semi-empirical models. They are obtained by combining one or more than one primary models, in order to obtain a general theory. They can be used as stress-strain-time or stress-strain-strain rate relationship that yield to solutions for creep as well as for relaxation.

### 3.2.1 Primary empirical relations

The empirical models described in this section are: (1) the semi-logarithmic law for creep, (2) Singh and Mitchell’s creep model, (3) Lacerda and Houston’s relaxation model, (4) Prevost’s model and (5) the strain rate approach.

**The semi-logarithmic law for creep**

This law is one of the first relationships proposed to describe the time dependent behaviour of soils. It is based on the results obtained from secondary compression of standard oedometer tests.

If the vertical strain or the void index is plotted versus the logarithm of time, the secondary compression phase can be approximately described with a strength line, defined by the secondary compression coefficient $C_{\alpha}$ (Figure 2.7). This coefficient can be described in different ways:

$$C_{\alpha e} = \frac{\Delta e}{\Delta \log(t)}$$  \hspace{1cm} (3.1)

with reference to the void index $e$, or:

$$C_{\alpha e} = \frac{\Delta \varepsilon_z}{\Delta \log(t)} = \frac{\Delta e}{(1 + e_i) \cdot \Delta \log(t)} = \frac{C_{\alpha e}}{1 + e_i}$$  \hspace{1cm} (3.2)
with reference to the vertical strain \( \varepsilon_z \); \( e_i \) is the initial void ratio and \( t \) is the time.

In the simplest form, the coefficient of secondary compression \( C_\alpha \) is assumed to be constant for one specific soil. This is an oversimplification of the volumetric confined creep. Several studies of the factors influencing \( C_\alpha \) have shown that the vertical effective stress \( \sigma_z' \), the preconsolidation pressure \( \sigma_{z,pc} \), the time \( t \) and other factors affect the secondary compression coefficient.

Within the framework of the semi-logarithmic law, two hypotheses are made on the secondary compression coefficient: (1) the concept of constant \( C_\alpha \) and (2) the concept of constant \( C_\alpha / C_c \).

**Concept of constant \( C_\alpha \)** The simplest assumption is that the secondary compression coefficient \( C_\alpha \) is constant for a given soil. The vertical strains can be evaluated as:

\[
\varepsilon_z = C_\alpha \varepsilon \cdot \log \left( 1 + \frac{t}{t_0} \right)
\]

where \( t_0 \) is some reference time.

One of the major difficulties to use Equation (3.3) is to decide when the creep deformation starts, i.e. determining the reference time \( t_0 \). This problem was discussed in Section 2.3.1.

The semi-logarithmic law is able to describe the behaviour of only primary creep, because Equation (3.3) predicts a gradual continuous reduction in the rate of compression.

The assumption of \( C_\alpha \) constant is in general too simplistic for a soil, but it is a good approximation in the normally consolidated range of stress.

**Concept of constant \( C_\alpha / C_c \)** As observed in Section 2.3.4, where the creep stress dependence is discussed with reference to the oedometer test, the ratio between the second compression coefficient \( C_\alpha \) and the compression index \( C_c \) has been found to be approximately linear over the entire applied stress range.

The compression index is defined as:

\[
C_{ce} = \frac{\Delta e}{\Delta \log(\sigma_z')}
\]

with reference to the void ratio, or:

\[
C_{ce} = \frac{\Delta \varepsilon}{\Delta \log(\sigma_z')} = \frac{C_{ce}}{1 + e_i}
\]

with reference to the vertical strain.
Mesri and Godlewski (1977) found that $C_\alpha$ depends on the applied effective stress $\sigma'_z$ and on the preconsolidation pressure $\sigma'_{z,pc}$. It was shown that both $C_\alpha$ and $C_c$ increase as the effective stress approaches the preconsolidation pressure, then reaches a maximum, to remains almost constant. Throughout these effective stress changes, the ratio $C_\alpha / C_c$ remains reasonably constant.

Combining Equation (3.3) with the assumption of constant $C_\alpha / C_c$, the vertical strain can be written as:

$$
\varepsilon_z = \frac{1}{m'} \cdot C_{c\varepsilon} \cdot \log \left( 1 + \frac{t}{t_0} \right)
$$

where $m'$ is the coefficient that defines the relationship between $C_\alpha$ and $C_c$.

The main advance of Equation (3.6), compared to Equation (3.3), is that the effect of the vertical stress $\sigma'_z$ is taken into account by the coefficient $C_c$.

**Concept by Yin** The main problem of the semi-logarithmic law is that the deformations tend to infinity when time tends to infinity. This behaviour is unrealistic and tends to overestimate the long term behaviour of soils. For this reason Yin (1999) proposed a new formulation of the semi-logarithmic law, in which the secondary compression coefficient $C_\alpha$ changes with time. The mathematical expression of this law is:

$$
\varepsilon_z = \frac{\psi}{\nu} \ln \left( 1 + \frac{t}{t_0} \right)
$$

where $\psi / \nu$ is defined by:

$$
\frac{\psi}{\nu} = \frac{\psi_0}{1 + \left( \frac{\psi_0}{\varepsilon_\infty} \right) \ln \left( 1 + \frac{t}{t_0} \right)}
$$

where $\nu = 1 + \epsilon$ is the specific volume, $t_0$ is the reference time, $\psi_0$ a model parameter and $\varepsilon_\infty$ the creep strain when time tend to infinity.

Equation (3.7) corresponds to the traditional logarithmic law (3.3) when the ratio $\psi / \nu$ is constant. In the general case the ratio $\psi / \nu$ decreases with time.

**Observations** The semi-logarithmic law for creep, and its three modifications, can describe the time dependent behaviour of soils under oedometer boundary conditions and constant vertical loading conditions. The main problem of this empirical law is to define the reference time $t_0$ or, in other words, the time that defines the onset of creep deformations.
Singh and Mitchell’s creep model

If there is no evidence of the tertiary phase during a creep process, the evolution of deferred strain rates with time can be described in first approximation with a straight line into a \( \log(\dot{\varepsilon}) - \log(t) \) diagram. The slope of this strain line is defined by the parameter \( m \) give by Singh and Mitchell (1968):

\[
m = \frac{\Delta \log(\dot{\varepsilon})}{\Delta \log(t)}
\]

Based on this observation Singh and Mitchell (1968) proposed a simple three-parameter phenomenological equation that may be used to describe the strain rate-time relation of clayey soils when subjected to constant stress:

\[
\dot{\varepsilon} = A \cdot e^{\bar{\alpha} \cdot \bar{q} \cdot \left(\frac{t_0}{t}\right)^m}
\]

where \( \bar{\alpha} = \alpha \cdot q_{\text{max}} \) and \( \bar{q} = q / q_{\text{max}} \). The parameter \( A \) reflect soil composition, structure and stress history; the parameter \( \alpha \) indicates the stress intensity effect on the creep rate and the parameter \( m \) controls the rate at which the strain rate decreases with time.

Integrating Equation (3.10) a general relationship between time and axial strain may be obtained. This integration yields two solutions, one for \( m \) equal to 1 and one for \( m \) different from 1. When \( m = 1 \) the integration yields a solution where the axial strain varies linearly with the logarithm of time. In the more general case, where \( m \neq 1 \), there is a non-linear relationship between the

![Figure 3.1: Creep curves predicted by the Singh and Mitchell’s model for \( m = 1 \), \( m < 1 \) and \( m > 1 \): (a) strain versus time and (b) strain versus logarithm of time](image-url)
axial strain and the logarithm of time. The axial creep is described by a power function. When time tends to infinite the strains tend to a constant value if \( m > 1 \), or tend to infinity if \( m < 1 \). The creep curves are shown in Figure 3.1.

**Observations** The Singh and Mitchell’s model is able to represent quite well the development of deferred strains during creep tests, both in the case of strain tending to an asymptotic value (fading creep) and in the case of strain tending to infinity (non-fading creep), for time tending to infinity. This different behaviour is governed by the value of the parameter \( m \): if \( m > 1 \) the creep is fading, if \( m < 1 \) the creep is non-fading. The main disadvantage of this model is to be limited only to oedometer and constant loading conditions and to represent correctly only the first phase loading (no secondary or tertiary creep).

**Lacerda and Houston’s relaxation model**

The main purpose of the Lacerda and Houston’s relaxation model (Lacerda and Houston, 1973) is to describe the time dependent behaviour of clay during stress relaxation tests, by means of constitutive relationships whose parameters can be obtained from standard creep tests. In order to describe the relation between stress relaxation and creep parameters, the three parameters constitutive law of Singh and Mitchell (1968) given in Equation (3.10) was assumed.

As reported in Section 2.4.2, experimentally the decrease of the deviatoric stress \( q \) with the logarithm of time can be approximated with a straight line, with the exception of the first phase of test where the trend is highly non linear. In order to approximate this behaviour a linear relationship between the deviatoric stress \( q \) and the logarithm of time, after an initial time period \( t_0 \), was assumed:

\[
\begin{align*}
\frac{q}{q_0} &= 1 \quad \text{for} \quad t \leq t_0 \\
\frac{q}{q_0} &= 1 - s \cdot \log \left( \frac{t}{t_0} \right) \quad \text{for} \quad t > t_0
\end{align*}
\]  

where \( t_0 \) is the initial time at the beginning of stress relaxation, which depends on soil type and strain rate, and \( s \) is the slope of the relaxation curve in the \( \frac{q}{q_0} - \log(t) \) diagram. Figure 3.2 depicts the assumptions of the model.

The slope \( s \) is related to the parameters \( \bar{\alpha} \) and \( m \) of Equation (3.10) in the following way:

\[
s = \frac{\Psi}{q_0} \quad \text{where} \quad \Psi = \frac{2.3 \cdot (1 - m)}{\bar{\alpha}}
\]  

It should be noted that Equation (3.12) is established for \( m < 1 \) which corresponds to the case of non-fading creep. Equations (3.11) and (3.12) are
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Figure 3.2: Stress relaxation behaviour for different strain rate values and Lacerda and Houston’s model representation

obtained by inverting the Equation (3.10) for creep, as shown by Lacerda and Houston (1973).

Observations This model is limited to one-dimensional conditions, because it is directly derived from the one-dimensional model of Singh and Mitchell.

The expression (3.11) predicts a non-fading stress relaxation, i.e., during a relaxation test the deviatoric stress decrease until zero. This behaviour is not correct, as reported by various authors which observed that experimentally the deviatoric stress does not decrease until zero but reaches a final relaxed value.

Prevost’s relaxation model

Prevost (1976) proposed a constitutive relationship in order to describe the behaviour of saturated clays under undrained triaxial relaxation test conditions. In the case where the initial stress state prior to relaxation is reached by shearing at a constant strain rate, the relationship proposed is:

\[
q(\varepsilon_{1,0}, t) = q(\varepsilon_{1,0}, t_0) - [q(\varepsilon_{1,0}, t_0) - q(\varepsilon_{1,0}, 0)] \cdot \tanh\left[b \cdot \ln\left(\frac{t}{t_0}\right)\right]
\]  

(3.13)

where \(q(\varepsilon_{1,0}, t)\) is the deviatoric stress acting at a fixed axial strain \(\varepsilon_{1,0}\) and time \(t > t_0\); \(q(\varepsilon_{1,0}, t_0)\) is the deviatoric stress acting at the beginning of stress relaxation and reached by shearing the specimen with a constant strain rate \(\dot{\varepsilon}_1\) in a time \(t_0\), that is \(\varepsilon_{1,0} = \dot{\varepsilon}_1 \cdot t_0\); \(q(\varepsilon_{1,0}, 0)\) is the deviatoric stress at a strain \(\varepsilon_{1,0}\) in a “static” compression test (\(\dot{\varepsilon}_1\) close to zero); \(b\) is a constitutive parameter and \(t_0\) is the time at which the relaxation starts. The trend of the deviator stress given by Equation (3.13) is depicted in Figure 3.3.
3.2. Empirical models

**Figure 3.3:** Deviatoric stress versus logarithm of time during stress relaxation for the Prevost’s relaxation model

**Observations** Contrarily to the Lacerda and Houston’s model, that considers a linear relationship, the Prevost’s model describes a non-linear relationship into the $q - \log(t)$ diagram. Another important difference is that the Prevost’s model considers final relaxed state of deviatoric stress (or static stress state) when the time tends to infinity, in contrast with the Lacerda and Houston’s model that assumes that the deviatoric stress never reaches a final relaxed state but decreases until zero. The final relaxed state is defined by $q(\varepsilon_{1,0}, 0) / q(\varepsilon_{1,0}, t_0)$, and this is therefore an input parameter of the model. The main disadvantage of the model is that it is limited to the one-dimensional case and cannot successfully be extended to a general three-dimensional case.

**Strain rate approach**

The main assumption in the strain rate approach (Leroueil *et al.*, 1985) is the existence of a unique relationship between the current state of effective stress $\sigma'_z$ and strain $\varepsilon_z$ for a given constant strain rate $\dot{\varepsilon}_z$. It is suggested that the $(\sigma'_z, \varepsilon_z, \dot{\varepsilon}_z)$ relationship can be described completely by two equations, one giving the variation of the preconsolidation pressure with the strain rate:

$$\sigma'_{z,pc} = f(\dot{\varepsilon}_z) \quad (3.14)$$

and the other presenting the effective stress-strain relationship normalized to the preconsolidation pressure:

$$\frac{\sigma'_z}{\sigma'_{z,pc}} = g(\varepsilon_z) \quad (3.15)$$
Equation (3.14) and Equation (3.15) are sketched in Figure 3.4. The normalized stress-strain curve represents the response of the soil to loading while the curve which describes the variation of the preconsolidation pressure with the strain rate represents the aptitude of clay to creep.

Once the two relationships are known for the given soil, any stress-strain-strain rate relationship for the soil may easily reconstructed.

Combining Equations (3.14) and (3.15) it is possible to obtain a general form of any solution:

$$\dot{\varepsilon}_z = f^{-1}\left(\frac{\sigma'_z}{g(\varepsilon_z)}\right)$$

(3.16)

Figure 3.4: Stress-strain-strain rate relationship for the strain rate approach: (a) variation of the preconsolidation pressure with the stain rate; (b) normalized effective stress-strain relation and (c) stress-strain curves obtained at different strain rate
Observations  This model permits to describe both creep and relaxation in one-dimensional tests. It is important to notice that the above relations must be written by decomposing the total strain in an elastic and in a viscoplastic part in order to describe correctly a stress relaxation test.

A limitation of this model is that the concept is developed mainly from observations in the normally consolidated range, which gives poor predictions in the heavily consolidated range where the elastic strains are relatively significant.

Observations on the primary empirical relations

The primary empirical relationship are influenced by the classical understanding of the time dependent phenomena of soils and, particularly, by the “correspondence principle” which states that all the different aspects of time dependence, as creep, stress relaxation, rate dependence and accumulated effects are considered to be due to the same basic mechanism. Therefore one phenomenon can be derived by the observation of another phenomenon and vice versa. According to this principle in the primary empirical relation there is not a stand-alone model for relaxation, but all relaxation models are derived from the model proposed for creep.

The fundamental distinction between the primary models is whether they are time or strain hardening. The time hardening models are characterized by a relation in which the time enter directly as an hardening parameter:

\[ \varepsilon^c = f(\sigma) \cdot g(t) \quad \text{or} \quad \dot{\varepsilon}^c = f(\sigma) \cdot g(t) \]  

(3.17)

where \( \varepsilon^c \) and \( \dot{\varepsilon}^c \) are respectively the creep strain and strain rate. For example, the model by Singh and Mitchell can be categorized as a time hardening model. On the other hand, the functional relation for a strain hardening model can be written as:

\[ \dot{\varepsilon}^c = f(\sigma) \cdot g(\varepsilon^c) \]  

(3.18)

The strain rate approach is a clear example of a strain hardening model.

3.2.2 Secondary semi-empirical models

The secondary semi-empirical models presented in this section are: (1) Kavazanjian and Mitchell’s approach, (2) Tavena’s approach, (3) Bjerrum’s model and (4) Yin and Graham’s model.

Kavazanjian and Mitchell’s approach

One of the first attempts to develop a multi-axial stress-strain-time constitutive model was proposed by Kavazanjian and Mitchell (1977).
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The main assumption of this model is to split the total strain into an instantaneous and in a delayed part. The delayed strains can be treated separately in a volumetric and in a deviatoric components.

The delayed volumetric component $\dot{\varepsilon}_p$ can be evaluated by using the semi-logarithmic law for creep discussed in Section 3.2.1. Therefore it is possible to write:

$$\dot{\varepsilon}_p = \frac{C_{\alpha \varepsilon}}{\ln(10)} \cdot \frac{1}{t}$$  \hspace{1cm} (3.19)

where $C_{\alpha \varepsilon}$ the secondary compression coefficient with reference to the vertical strain.

The deviatoric component $\varepsilon_q$ of the deferred deformation can be calculated by using the Singh and Mitchell’s model presented in Section 3.2.1. If the axial strain rate of Equations (3.10) is taken as the strain rate along the first principal axis $\varepsilon_1$, Equation (3.10) can be written as:

$$\dot{\varepsilon}_1 = A \cdot e^{\alpha q} \cdot \left( \frac{t}{t_0} \right)^m$$  \hspace{1cm} (3.20)

The deviatoric strain rate $\dot{\varepsilon}_q$ can be calculated indirectly from the axial strain in triaxial conditions:

$$\dot{\varepsilon}_q = \dot{\varepsilon}_1 - \frac{\dot{\varepsilon}_p}{3}$$  \hspace{1cm} (3.21)

where $\dot{\varepsilon}_p$ and $\dot{\varepsilon}_1$ are given by Equation (3.19) and Equation (3.20) respectively.

**Tavena’s approach**

Tavenas et al. (1978) assumed to split the delayed deformations into a volumetric $\varepsilon_p$ and a deviatoric $\varepsilon_q$ part, that can be described using the Singh and Mitchell’s relationship:

$$\dot{\varepsilon}_p = B \cdot f(\sigma_{ij}') \cdot \left( \frac{t}{t_0} \right)^m$$  \hspace{1cm} (3.22)

$$\dot{\varepsilon}_q = A \cdot g(\sigma_{ij}') \cdot \left( \frac{t}{t_0} \right)^m$$  \hspace{1cm} (3.23)

where $f(\sigma_{ij}')$ and $g(\sigma_{ij}')$ are function of the effective state of stress $\sigma_{ij}'$, A and B are soil properties and $m$ is a constitutive parameter that governs the rate at which the strain rate decreases. It is important to notice that the same power parameter $m$ is assumed to describe both the volumetric and the deviatoric delayed strain, which is not likely the general pattern.

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3.2. Empirical models

Based on the shape of the contour lines for equal strain rate, Tavenas et al. (1978) suggest that the stress functions \( f(\sigma'_{ij}) \) and \( g(\sigma'_{ij}) \) should be expressed in terms of the limit state surface, also denoted as yield surface.

The ratio between the volumetric and the deviatoric creep rates could be expressed as a function of the current effective stress state \( \sigma'_{ij} \) only (Sekiguchi, 1985):

\[
\frac{\dot{\varepsilon}_p}{\dot{\varepsilon}_q} = \frac{f(\sigma'_{ij})}{g(\sigma'_{ij})} = h(\sigma'_{ij})
\]  

(3.24)

in which \( h(\sigma'_{ij}) \) is a material function.

**Bjerrum’s approach**

Bjerrum presented a concept for settlement analysis of normally and lightly over-consolidated clays that includes the logarithmic law.

The Bjerrum theory is based on the observations from oedometer compression tests that indicate that there is not a single stress-strain compression curve in the \( e - \log(\sigma'_{z}) \) diagram, but a family of parallel curves, called “time lines”, which correspond to a different duration of the applied load (Figure 3.5). An important characteristic is that the value of the preconsolidation pressure \( \sigma'_{z,pc} \)

![Figure 3.5: Geological history and compressibility of a young and aged normally consolidated clay after Bjerrum approach](Image)

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depends on the considered time line. The different time lines represent a series of equilibrium relationships after different period of sustained loading.

In Figure 3.5 the oedometer compression curves for a “young” and an “aged” normally consolidated clay are depicted. The young NC clay shows a preconsolidation pressure $\sigma'_{z,pc0}$ equal to the active vertical effective stress $\sigma'_{z,0}$. If the vertical load is left acting on the young NC clay for several years, the clay compresses under constant effective stress with time (creep) which is denoted as delayed compression or secondary compression. Afterwards, the aged NC clay follows the lower curve and the measured preconsolidation pressure has increased to $\sigma'_{z,pc1}$. This means that the reduction in the void ratio that occur during secondary compression leads to a more stable clay structure and therefore a higher preconsolidation pressure.

Garlanger (1972) has modelled the approach proposed by Bjerrum, in terms of the well known recompression, compression and secondary compression indices $C_{re}$, $C_{ce}$ and $C_{ae}$ respectively, with reference to a logarithmic function. The main assumption is to split the change in the void ratio into an elastic part ($e^e$), an elastoplastic part ($e^{ep}$) and in a delayed creep part ($e^c$).

$$
e = e_0 - e^e - e^{ep} - e^c$$

$$
e = e_0 - C_{re} \log \frac{\sigma'_{z,pc}}{\sigma'_{z,0}} - C_{ce} \log \frac{\sigma'_{z}}{\sigma'_{z,pc}} - C_{ae} \log \frac{t_0 + t}{t} \quad (3.25)$$

where $e$ is the void ratio, $e_0$ the initial void ratio, $\sigma'_{z,pc}$ the vertical preconsolidation pressure, $\sigma'_{z,0}$ the initial vertical effective in situ stress, $\sigma'_{z}$ the current vertical effective stress, $t$ the time and $t_0$ the reference time.

The amount of the vertical preconsolidation pressure can be derived by means of Equation (3.25) as illustrated in Figure 3.6. The initial condition is a young NC clay, with a initial void ratio $e_0$, an initial time $t_0$ and initial stress state $\sigma'_{z,0}$ equal to the preconsolidation pressure $\sigma'_{z,pc0}$. The final condition is an aged NC clay, with a void ratio $e_t$, a time $t$, and a preconsolidation pressure $\sigma'_{z,pct}$. The relation for age dependence is given by:

$$
\frac{\sigma'_{z,pct}}{\sigma'_{z,pc0}} = \left( \frac{t}{t_0} \right)^{\frac{C_{ae}}{C_{ae} - C_{re}}} \quad (3.26)
$$

The ratio $C_{ae} / (C_{ce} - C_{re})$ is similar to the parameter $m'$ if the recompression index $C_{re}$ is neglected.
3.2. Empirical models

Yin and Graham’s model

Yin and Graham (1996) proposed a series of one-dimensional models, based on the Bjerrum’s approach, described above, and on the strain rate approach, described in the previous section. These models allow to describe the time dependent behaviour of both NC and OC clays under various test conditions: relaxation tests, constant strain rate tests and constant strain rate tests.

Yin and Graham’s model is based on the concepts of: (1) equivalent time, (2) time line, (3) reference time line, (4) instant time line and (5) limit time line. These are illustrated in Figure 3.7.

- Equivalent time and time line. Yin and Graham (1996) defines time lines as lines having the same equivalent time \( t_e \). The equivalent time \( t_e \) is defined as the time needed to creep from the reference time line, for which \( t_e = 0 \), to the current value of the vertical strain \( \varepsilon_z \) and the vertical effective stress \( \sigma'_z \). In a classic oedometer test, in the normally consolidated range the equivalent time is equal to the duration of each load increment but in the over-consolidated range it can differ considerably. The equivalent time is related to a unique creep strain rate, with the minimum value associated to the larger strain rates. Equivalent time lines below the reference time line are in the positive range, while lines above the reference line are in the negative range.
• Reference time line: is defined as the equivalent time line at which the equivalent time \( t_e \) is equal to zero.

• Instant time line: this line defines the instantaneous loading response of the soil. The strains are assumed to be purely elastic.

• Limit time line: this line defines the limit in the \( \varepsilon_z - \sigma'_z \) space, above which the behaviour is time independent. It can be obtained with an equivalent time tending to infinite or, likewise, with a vertical strain rate tending to zero.

\[
\dot{\varepsilon}_z = \frac{\kappa}{\nu} \cdot \frac{1}{\sigma'_z} \cdot \dot{\sigma}'_z + \frac{\psi}{\nu \cdot t_0} \cdot \exp \left[ - \frac{(\varepsilon_z - \varepsilon_{z0})}{\psi} \cdot \frac{\nu}{\sigma'_z \sigma'_{z0}} \right] \tag{3.27}
\]

where \( \varepsilon_{z0} \) is the initial strain, \( \sigma'_{z0} \) the initial effective stress, \( \nu \) the specific volume, \( \kappa \) a material parameter that describes the elastic stiffness of the soil, \( \lambda \)

**Figure 3.7:** Definition of instant time line, reference time line, limit time line and positive and negative time lines according to Yin and Graham’s model

Yin and Graham proposed two types of general elasticviscousplastic models, one formulated by means of logarithmic function and one by means of power function.

The general stress-strain relationship for the logarithmic formulation is given by:
3.3 Rheological models

an elasticplastic material parameter, $\psi$ a creep parameter and $t_0$ an intrinsic time parameter.

The general stress-strain relationship for power formulation is given by:

$$\dot{\varepsilon}_z = a_2 n_1 \left( \frac{\sigma'_z}{\sigma'_u} - \frac{\sigma'_{z0e}}{\sigma'_u} \right)^{n_1-1} \dot{\sigma}'_z + (f^{ep}_\infty - f^{ep}_0) \frac{n_3}{t_0} \left( 1 - \frac{\varepsilon_z - f^{ep}_0}{f^{ep}_\infty - f^{ep}_0} \right)^{n_3+1} \quad (3.28)$$

where $f^{ep}_0$ indicates the function of the reference time line, $f^{ep}_\infty$ indicates the function of the limit time line, $\sigma'_u$ the stress unit, $a_2$, $n_1$, $n_3$, $\sigma'_{z0e}$ model parameters and $t_0$ intrinsic time parameter.

The general differential Equations (3.27) and (3.28) can be solved in order to obtain solutions for creep, stress relaxation, constant strain rate or constant stress rate tests.

Observations on the secondary semi-empirical models

The correspondence principle holds true for the secondary semi-empirical models. Generally a creep law is introduced in a time independent model, obtaining a general relationship that can provide solutions for creep, stress relaxation, constant strain rate and constant stress rate tests. It should be noticed that the secondary semi-empirical models rely on the same hardening methods, time hardening and strain hardening, discussed in the previous section.

3.3 Rheological models

The rheological models have been presented in literature in order to describe the time dependent behaviour of metals and steels (at high temperature) and fluid, and have been extended to geomaterials. They are generally formulated with reference to one-dimensional conditions and supply directly the fundamental relationships that govern the time dependency, either in a differential or in a closed form. They can be subsequently extended to the three-dimensional case.

The main advantage of the rheological models is to supply a quite simple and intuitive functioning mechanism.

They are usually divided into three categories as follows:

- Analogical models or mechanical rheological models. The constitutive relations are constructed by combining, in series or in parallel, different elementary material models, such as Hookean, Saint-Venant’s and Newtonian material models.
• Engineering theories of creep. General theories for determining the delayed behaviour of metal or concrete. They supply directly the mathematical formulation of the relationship that describes the time dependence.

• The hereditary approach, also known as the method of the integral representation. In this approach the time dependent creep strain or stress is defined by a “creep” or “relaxation” function, which is a hereditary (memory) function describing the history dependence of strain or stress.

The basic principle of this category is briefly discussed in the following subsections with reference to uniaxial conditions.

3.3.1 Analogical models

Different analogical models have been proposed for the mathematical description of the stress-strain-time behaviour of soils. They are made up of combinations of elementary material models, such Hookean (schematized as a spring), Saint-Venant’s (schematized as a slider) and Newtonian (schematized as a viscous dashpot) models. The elementary material models can be combined in series or in parallel. The characteristics of these three elementary models are illustrated in Figure 3.8 with reference to the simple one-dimensional case.

Frequently the analogical models are formulated with reference to one-dimensional conditions by combining elementary models and are extended to the general three-dimensional case by subdividing the behaviour into a volumetric and a deviatoric component.

It is important to notice that the Hookean spring and the Newtonian viscous dashpot could also be highly non linear. Moreover, the Saint-Venant’s slider could also represent a plastic element governed by the general three-dimensional theory of plasticity.

Some examples of analogical models that have been found to give reasonable approximations of the behaviour of some soils under given loading conditions are shown in Figures 3.9 and 3.10. It is noted that in the Murayama and Shibata, and Christensen and Wu models the dashpots are non linear, with stress-flow rate response governed by rate process theory.

In the following sub-section only two models are briefly discussed in order to present the potential of the rheological analogical models.
3.3. Rheological models

Figure 3.8: Schematic representation of elementary material models: (a) Hookean linear elastic spring, (b) Newtonian viscous dashpot and (c) Saint-Venant’s slider. The superscript \( e \) and \( \eta \) denotes elastic and viscous respectively. \( E \) is the spring constant and \( \eta \) is the viscosity constant. The slider should be understood as an ideal plastic element that is inactive (locked) below a sudden threshold (yield) stress \( \sigma_y \); if the stress \( \sigma \) exceeds \( \sigma_y \) the slider is unlocked and plastic deformations are allowed.

Figure 3.9: Some analogical models proposed for characterization of the time dependent behaviour of soil.
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(a) Bingham

(b) Christensen and Wu

(c) Murayama ans Shibata

(d) CVISC

(e) Cividini and Gioda

(f) Cividini and Gioda simplified

Figure 3.10: Some analogical models proposed for characterization of the time dependent behaviour of soil
3.3. Rheological models

CVISC model

The CVISC model (ITASCA, 2006) is an analogical model which couples, in series, the Burgers viscoelastic model with a plastic flow rule, based on the Mohr-Coulomb yield criterion, as shown in Figure 3.11.

![Sketch of the CVISC model](image)

The main assumption of this model is to split the mechanical behaviour of the material into volumetric and deviatoric components. In terms of strain one writes:

\[
\varepsilon_p = \varepsilon_{mm} \quad \text{and} \quad \varepsilon_p = \varepsilon^e_p + \varepsilon^p_p \quad (3.29)
\]

\[
e_{ij} = \varepsilon_{ij} - \frac{1}{3} \cdot \delta_{ij} \cdot \varepsilon_{mm} \quad \text{and} \quad e_{ij} = e^{ve}_{ij} + e^{P}_{ij} \quad (3.30)
\]

The volumetric behaviour is only elastoplastic and is governed by the linear elastic law and the plastic flow rule (Figure 3.11.a), while the deviatoric behaviour is viscoelastoplastic and is driven by the Burgers model and the same plastic flow rule (Figure 3.11.b). This means that the viscoelastic strains are deviatoric and depend only on the deviator stress \( s_{ij} \); instead the plastic strains are both deviatoric and volumetric and depend on the total stress \( \sigma_{ij} \) in accordance with the chosen flow rule.
The viscoelastic part can be described by the following relations, which hold true for the Kelvin element (superscript \(^K\)) and for the Maxwell element (superscript \(^M\)) respectively. These relations form, in series, the Burgers model.

\[
e_{ij}^{ve} = e_{ij}^{veK} + e_{ij}^{veM}
\]

**Kelvin element:**
\[
s_{ij} = 2 \cdot G^{K} \cdot e_{ij}^{veK} + 2 \cdot \eta^{K} \cdot \dot{e}_{ij}^{veK}
\]

**Mawell element:**
\[
\dot{e}_{ij}^{veM} = \frac{\dot{s}_{ij}}{2G^{M}} + \frac{\dot{s}_{ij}}{2\eta^{M}}
\]

The elastic strain component of the volumetric behaviour is described by the relation:

\[
p = K \cdot e_{p}^{e}
\]

The plastic strains follow the general flow rule of plasticity:

\[
\dot{\varepsilon}_{ij}^{p} = \lambda \cdot \frac{\partial g}{\partial \sigma_{ij}}
\]

where \(\lambda\) is the plastic multiplier and \(g\) the plastic potential, with the same shape of the Mohr-Coulomb yield criterion, but controlled by the dilation angle \(\psi\).

The model is characterised by nine parameters as follows: one elastic property (bulk modulus \(K\)), four viscoelastic properties (Maxwell dynamic viscosity \(\eta^{M}\) and shear modulus \(G^{M}\), Kelvin dynamic viscosity \(\eta^{K}\) and shear modulus \(G^{K}\)) and four plastic properties (cohesion \(c\), friction angle \(\phi\), uniaxial tensile strength \(\sigma_{t}\) and dilation angle \(\psi\)).

The constitutive parameters can be derived by means of analytical and numerical fitting of experimental data. The elastic and plastic parameters can be determined from classical deformation and strength tests following a standard procedure. The viscoelastic parameters can be derived from triaxial tests (creep, relaxation, constant strain rate tests). It is however more reliable to use creep tests.

**Creep** In the case of creep tests, the main assumption is that the stress level is lower than the yield strength: without the plastic component the CVISC model reduces to the Burgers model. For a cylindrical specimen subjected to axial and radial stresses, \(\sigma_{a}\) and \(\sigma_{r}\), the induced axial and radial viscoelastic strains, \(\varepsilon_{a}\) and \(\varepsilon_{r}\), are:
3.3. Rheological models

\[ \varepsilon_a = \frac{p}{3K} + \frac{q}{3G^M} + \frac{q}{3\eta^M}t + \frac{q}{3G^k} \left[ 1 - \exp \left( -\frac{G^K}{\eta^k} t \right) \right] \] (3.36)

\[ \varepsilon_r = \frac{p}{3K} - \frac{q}{6G^M} - \frac{q}{6\eta^M}t - \frac{q}{6G^k} \left[ 1 - \exp \left( -\frac{G^K}{\eta^k} t \right) \right] \] (3.37)

Both these relations allow one to determine the required parameters by means of numerical fitting (Bonini et al., 2007; Debernardi, 2004). It is observed that Equation (3.36) is more suitable for this purpose, because the axial strain \( \varepsilon_a \) is in general measured in a more accurate and reliable manner.

**Cividini and Gioda’s simplified model**

The Cividini and Gioda’s simplified model (Gioda and Cividini, 1996) comprises a Kelvin-Voight element in series with a Bingham element. The plastic yield criterion of the slider of the Bingham element is defined according to the Mohr-Coulomb yield criterion. As for the CVISC model the main assumption is to split the mechanical behaviour of the soil into a volumetric and a deviatoric component. Also in this model the volumetric behaviour is assumed to be only elastoplastic and is governed by the linear elastic law and the plastic flow rule (Figure 3.12), while the deviatoric behaviour is viscoelastoplastic.

![Figure 3.12: Sketch of the Cividini and Gioda’s simplified model](image)

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\[ \varepsilon_p = \varepsilon_{mm} \quad \text{and} \quad \varepsilon_p = \varepsilon_e^p + \varepsilon_p^p \quad (3.38) \]
\[ \varepsilon_{ij} = \varepsilon_{ij} - \frac{1}{3} \cdot \delta_{ij} \cdot \varepsilon_{mm} \quad \text{and} \quad \varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^{ve} + \varepsilon_{ij}^p \quad (3.39) \]

The three elements of the rheological model are described by:

**Elastic element:**
\[ \sigma_{ij} = K \cdot \varepsilon_{p}^e + 2 \cdot G \cdot \varepsilon_{ij}^e \quad (3.40) \]

**Kelvin element:**
\[ s_{ij} = 2 \cdot G^K \cdot \varepsilon_{ij}^{ve} + 2 \cdot \eta^K \cdot \dot{\varepsilon}_{ij}^{ve} \quad (3.41) \]

**Bingham element:**
\[ s_{ij}^{BD} = \eta^B \cdot \dot{\varepsilon}^p \quad (3.42) \]

where \( s_{ij}^{BD} \) is the portion of the deviatoric stress carried by the viscoplastic dashpot of the Bingham element. This can be determined by introducing the total stress carried by the plastic slider \( \sigma_{ij}^{PS} \), which includes the entire hydrostatic stress and fulfils the viscoplastic limit condition:

\[ s_{ij}^{BD} = \sigma_{ij} - \sigma_{ij}^{PS} \quad (3.43) \]
\[ \sigma_{mm}^{PS} = \sigma_{mm} \quad (3.44) \]

where the stress tensor \( \sigma_{ij}^{PS} \) must satisfy the consistency condition:

\[ f(\sigma_{ij}^{PS}) = 0 \quad (3.45) \]

In addition the plastic flow rule requires that:

\[ \dot{\varepsilon}_{ij}^p = \lambda \cdot \frac{\partial g}{\partial \sigma_{ij}^{PS}} \quad (3.46) \]

where \( \lambda \) is the plastic multiplier and \( g \) the plastic potential, with the same shape of the Mohr-Coulomb yield criterion, but controlled by the dilation angle \( \psi \).

The model is characterised by nine parameters as follows: two elastic properties (bulk modulus \( K \) and shear modulus \( G \)), three viscoelastic properties (Bingham dynamic viscosity \( \eta^B \), Kelvin dynamic viscosity \( \eta^K \) and shear modulus \( G^K \)) and four plastic properties (cohesion \( c \), friction angle \( \phi \), uniaxial tensile strength \( \sigma_t \) and dilation angle \( \psi \)).
Creep  In the case of creep test if the stress level is below the failure surface defined by the Mohr-Coulomb yield criterion, the model is reduced to a Kelvin-Voight model and the axial and radial strains, $\varepsilon_a$ and $\varepsilon_r$ can be determined using:

$$
\varepsilon_a = \frac{p}{3K} + \frac{q}{3G} + \frac{q}{3Gk} \left[ 1 - \exp \left( -\frac{G^K}{\eta_k} t \right) \right] \quad (3.47)
$$

$$
\varepsilon_r = \frac{p}{3K} - \frac{q}{6G} - \frac{q}{6Gk} \left[ 1 - \exp \left( -\frac{G^K}{\eta_k} t \right) \right] \quad (3.48)
$$

If the stress level induces sliding of the Bingham slider, a closed-form solution of time dependent deformations cannot be found and a numerical solution procedure is required. The Cividini and Gioda’s simplified model permits to describe correctly both the primary and the secondary phase of creep.

Sterpi and Gioda (2007) have recently modified this model in order to take into account also the tertiary phase of creep. This has been done by modifying the parameters of the Bingham element from a peak value to a residual value. The chosen shape of decay of the parameters $c$, $\phi$, $\psi$ and $\eta^B$ with the deviatoric viscoplastic strain is depicted in Figure 3.13.

**Figure 3.13:** Variation of the viscoplastic parameters with the 2nd invariant of deviatoric viscoplastic strains for the Sterpi and Gioda’s model.

**Observations on the analogical models**

Rheological analogical models are useful conceptually to aid in recognition of elastic, plastic and viscous components of deformation. They are helpful for visualization by analogy of viscous flow that accompanies time dependent changes of structures to a more stable state. Mathematical relationships can
be developed in a straightforward manner for the description of creep, stress relaxation, steady state deformation, etc., in terms of the model constants. In most cases, these relationships are complex and necessitate the evaluation of several parameters that may not be valid for different stress intensities or soil states.

3.3.2 Engineering theories of creep

Most of the engineering theories of creep have been proposed with reference to the experimental observations on metals, steel, concrete and ice, where the applied loading is below the yield stress. They are phenomenological and there are obvious similarities between the structures of the empirical models presented in Section 3.2 and the engineering theories of creep. However, the engineering theories of creep differ conceptually from the traditional way of dealing with creep in soils, because they are seen as creep theories for materials where the stress state is below the yield limit. For soils, the empirical models of creep have been originally formulated for plastic normally consolidated soils.

The engineering theories of creep can be distinguished in three categories: (1) the total strain model, (2) the time hardening model, and (3) the strain hardening model.

**Total strain model**

The main assumption of the total strain model is to subdivide the total strain into an elastic component (superscript \( \varepsilon^e \)) and into a deferred creep component (superscript \( \varepsilon^c \)), as follows:

\[
\varepsilon = \varepsilon^e + \varepsilon^c
\]  

(3.49)

For a standard creep test, where the stress \( \sigma \) is applied instantaneously, the creep strain component \( \varepsilon^c \) is given by a straightforward relationship of the applied stress \( \sigma \) and time \( t \) (loading history):

\[
\varepsilon^c = f(\sigma) \cdot g(t)
\]  

(3.50)

where \( f \) and \( g \) are functions. The relationship between creep strain and stress is often modelled by a power function: the creep strain depends non linearly on the applied stress, in contrast to the analogical models for which this relationship is assumed to be linear. Frequently also the relationship between creep strain and time in assumed to be a power function. Several other expressions are available, for instance exponential or hyperbolic functions.
3.3. Rheological models

Time hardening model

In order to consider the variation of the stress, in the time hardening model, the constitutive relationship is formulated in differential form, between creep strain rate $\dot{\varepsilon}^c$, stress $\sigma$ and time $t$:

$$\dot{\varepsilon}^c = f(\sigma) \cdot g(t)$$  \hspace{1cm} (3.51)

where generally $f$ and $g$ are non linear functions. In this equation time can be considered as an hardening parameter, hence the name “Time hardening model”. One of the shortcomings of this model, as well as the total strain model, is that the governing equations are not invariant with respect to the origin of time, because it is introduced in explicit form.

Strain hardening model

In the strain hardening model the mechanism that governs the hardening is assumed to be the accumulation of deferred creep strains or creep work. The constitutive relationship of the model is formulated in a differential form, between the creep strain rate, the applied stress and the creep strains:

$$\dot{\varepsilon}^c = f(\sigma) \cdot g(\varepsilon^c)$$  \hspace{1cm} (3.52)

where $f$ and $g$ are non linear functions. The creep strain $\varepsilon^c$ characterizes the state of the material and may therefore be viewed as an internal variable.

Observations on the engineering theories of creep

The concept of hardening in the engineering theories of creep is analogous to the concept of hardening of soils in the empirical models, described in Section 3.2. This is shown by the similarities between the mathematical formulations of the two models. The empirical models that are related in structure to the time and strain hardening models are listed in Table 3.1.

In the above, it is seen that the engineering theories of creep are based upon an a priori adopted creep relation. The ability to predict relaxation or constant strain rate conditions are secondary and rely on the existence of the correspondence principle. The solutions for the time hardening model are relatively simple, whereas the solutions for the strain hardening model become relatively complex.
Table 3.1: Correspondence between the engineering theories of creep and the empirical models

<table>
<thead>
<tr>
<th>Creep theory</th>
<th>Relation</th>
<th>Corresponding empirical model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total strain model</td>
<td>$\varepsilon^c = f(\sigma) \cdot g(t)$</td>
<td>Semi logarithmic law. The stress function $f$ is either constant in constant $C_\alpha$ approach, or varies with the confining stress in the constant $C_c/C_\alpha$ approach. The time function $g$ is the logarithmic function.</td>
</tr>
<tr>
<td>Time hardening model</td>
<td>$\dot{\varepsilon}^c = f(\sigma) \cdot g(t)$</td>
<td>Singh and Mitchell’s model. The stress function $f$ is an exponential function, and the time function $g$ is a power function with the exponent $m$.</td>
</tr>
<tr>
<td>Strain hardening model</td>
<td>$\dot{\varepsilon}^c = f(\sigma) \cdot g(\varepsilon^c)$</td>
<td>Strain rate approach. The general form of the strain rate approach states a unique relation between the stress, strain and strain rate.</td>
</tr>
</tbody>
</table>

### 3.3.3 Hereditary approach

The principle of the hereditary approach is that the current strain $\varepsilon(t)$ is obtained by integration over the entire loading history, i.e. integration over all infinitesimal stress changes on the current time $t$, hence, the name hereditary approach. The theory is developed for two cases. The simplest case is the hereditary approach based on linear viscoelasticity, which is a generalization of the analogaical approach. It is also possible to adopt the hereditary approach to non linear material behaviour corresponding to a generalization of the engineering theories of creep. The general opinion is that the hereditary approach is too complex to be incorporated in soil mechanics. This is due to the fact that the approach pays for its generality by a great number of experiments needed for calibration. Feda (1992) reported that 28 tests are needed to describe a uniaxial stress experiment by non linear hereditary theory, and for a triaxial state of stress about six times as many are required. However, it may be possible to reduce the number of tests for simple boundary conditions.

### 3.4 General theories

The general theories of elastoviscoplasticity represent the most advanced aspects of numerical modelling of time dependent behaviour of soils. These theories
describe not only the viscous effects but also the inviscid (time independent) behaviour of soils, and can include the most recent aspects of the research on plasticity. These theories are not limited to specific boundary and loading conditions, and can describe all possible stress paths and boundary conditions. They are often proposed in a general three-dimensional formulation, with the explicit goal to be implemented into a numerical analysis code, as finite element or finite difference methods codes. Rarely a closed form solution is proposed.

Elasticviscoplastic models can be divided into three classes: (1) elastoviscoplastic models based on the overstress theory of Perzyna, (2) elastoviscoplastic models based on the concept of a nonstationary flow surface and (3) others.

3.4.1 Perzyna’s overstress theory

The overstress theory was originally proposed by Perzyna (1966) and Olszak and Perzyna (1966), by combining the experimental results on the time dependent behaviour of metals, with the classical theory of elastoplasticity. This new theory was called elastoviscoplasticity and contains the theory of elastoplasticity as a specific case. Recently it has been extended to geomaterials.

Formulation

The overstress theory is based upon the existence of a “static surface” in the effective stress space, as described in the previous chapter with reference to the results of experimental investigations. According to this observation, a limit surface $f(\sigma_{ij}) = 0$, also called “yield surface”, is introduced in the effective stress space.

The key assumption of the overstress theory is that the point, which represents the effective state of stress $\sigma_{ij}$, can cross the yield surface and go out from the internal area, during a loading phase. In other words it is equivalent to suppress the consistence conditions of the classical theory of elastoplasticity, thus allowing the yield function $f$ to be positive or negative.

The yield surface defines two different fields in the effective stress space (Figure 3.14):

- An elastic field inside the yield surface ($f(\sigma_{ij}) < 0$). Inside this area the deformations are only elastic (no viscoplastic deformations develop) and they can be determined using the generalized Hooke’s law of elasticity:

$$ \dot{\varepsilon}^e_{ij} = C_{ijkl} \cdot \dot{\sigma}_{kl} $$ (3.53)

This field is completely coincident to the elastic field of elastoplasticity.
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Figure 3.14: Static yield surface and stress fields in the effective stress space for the Perzyna’s overstress theory

- An elasto-viscoplastic field external to the yield surface \( f(\sigma_{ij}) > 0 \). Inside this field the deformations are elasto-viscoplastic. It is possible to split the total strain rate tensor into an elastic and a viscoplastic component to give:

\[
\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^{vp}
\]

where the superscripts \( ^e \) and \( ^{vp} \) denote elastic and viscoplastic parts respectively.

While the elastic strain rate \( \dot{\varepsilon}_{ij}^e \) can be determined using the generalized Hooke’s law of Equation (3.53), the viscoplastic strain rate \( \dot{\varepsilon}_{ij}^{vp} \) can be evaluated by the following non-associated general flow rule:

\[
\dot{\varepsilon}_{ij}^{vp} = \gamma \cdot \Phi (F) \cdot \frac{\partial g}{\partial \sigma_{ij}}
\]

where \( \gamma \) is a fluidity parameter, \( \Phi (F) \) is the so-called viscous nucleus, \( F \) is the overstress function, \( g \) is the viscoplastic potential function and \( \sigma_{ij} \) the effective state of stress.

The overstress function \( F \) represents a measure of the distance between the point, which corresponds the current effective state of stress, and the yield surface (Figure 3.14), hence the name of overstress function. From a mechanical point of view the overstress function \( F \) would represent a measure of the unbalanced forces in the material.
3.4. *General theories*

The viscoplastic potential function $g$ defines the direction of the viscoplastic strain rate $\dot{\varepsilon}_{ij}^{vp}$ (i.e. the viscoplastic strain rate vector is perpendicular to the equi-potential surface $g(\sigma_{ij}) = \text{const}$, as shown in Figure 3.14), while the overstress function $F$ influences its modulus by means of the viscous nucleus $\Phi$.

**Hardening of the yield surface**

Generally the yield surface is not fixed, but can change shape and position as a consequence of the development of viscoplastic strains. In other words the yield surface can harden.

Frequently, the hardening of the yield surface is assumed to be governed by the viscoplastic deformations $\varepsilon_{ij}^{vp}$. The most used factors are the viscoplastic deviatoric strain $\varepsilon_{q}^{vp}$:

$$
\varepsilon_{q}^{vp} = \sqrt{\frac{2}{3} e_{ij}^{vp} \cdot e_{ij}^{vp}}
$$

(3.56)

where $e_{ij}^{vp}$ is the deviator viscoplastic strain tensor:

$$
e_{ij}^{vp} = \varepsilon_{ij}^{vp} - \frac{1}{3} \delta_{ij} \cdot \varepsilon_{mm}^{vp}
$$

(3.57)

or the viscoplastic work $W^{vp}$:

$$
W^{vp} = \int_{0}^{t} \sigma_{ij} \cdot \dot{\varepsilon}_{ij}^{vp} \cdot d\tau
$$

(3.58)

Perzyna (1966) suggests that the hardening of the yield surface can be described by decomposing the yield function $f$ into a component $\bar{f}$, which depends only on the effective state of stress $\sigma_{ij}$ and into a component $\kappa$, which depends only on the viscoplastic deviatoric strain $\varepsilon_{ij}^{vp}$, as follows:

$$
f = \frac{\bar{f}(\sigma_{ij})}{\kappa(\varepsilon_{ij}^{vp})} - 1
$$

(3.59)

**Viscous nucleus**

The choice of the viscous nucleus is very important in order to describe the time dependent response of a material. Generally the viscous nucleus is supposed to provide zero values for values of $F$ negative. This allows to have not viscoplastic deformations inside the static yield surface.

The viscous nucleus $\Phi$ must be a monotonous function of the overstress function $F$, thus the following relationship hold always true:
\[ \frac{\partial \Phi(F)}{\partial F} \geq 0 \quad (3.60) \]

In literature it is possible to find three different definitions of the viscous nucleus function:

- **Linear relationship:**
  \[ \Phi(F) = \frac{F}{F_0} \quad (3.61) \]
  where \( F_0 \) is the reference unity.

- **Power relationship:**
  \[ \Phi(F) = \left( \frac{F}{F_0} \right)^n \quad (3.62) \]
  where \( F_0 \) is the reference unity and \( n \) is a constitutive parameter. This assumption is frequently adopted (Nguyen-Minh, 1986) and allows to obtain a satisfactory fitting of the experimental results.

- **Exponential relationship:**
  \[ \Phi(F) = A \cdot \left\{ \exp \left[ \left( \frac{F}{F_0} \right)^n \right] - 1 \right\} \quad (3.63) \]
  where \( F_0 \) is the reference unity and \( A \) and \( n \) are constitutive parameters (Fodil et al., 1997).

**Observations on the overstress theory**

**Creep.** Consider a creep process initiated at a stress point lying outside the static yield surface \( (F > 0) \), as shown in Figure 3.15. If \( f \) is a non-hardening yield function, viscoplastic flow occurs and continues to occur at a constant rate (i.e. the distance between the static yield surface and the stress point remains constant with time). If \( f \) is a hardening yield function, viscoplastic flow occurs at a decreasing rate, because, as viscoplastic strain accumulates, the static yield surface \( f \) changes in such way that \( F \rightarrow 0 \), thus \( \dot{\varepsilon}_{ij}^{vp} \rightarrow 0 \). That is, the distance between \( f \) and the stress point decreases with a decreasing rate. In other words, the static yield surface moves out with time, and finally, after an infinite time passes through the stress point.
3.4. General theories

Consider now a stress state that lies outside the static yield surface \((F > 0)\) and a total strain rate equal to zero (Figure 3.16). This corresponds to a stress relaxation process, and implies, according to Equation (3.54), that elastic strains are equal in magnitude to the viscoplastic strains but opposite in direction. As viscoplastic strain develops the stress decreases. This implies that the static yield function \(f\) hardens and the stress point moves toward the static yield surface. After an infinite period of time the static yield function passes through the stress point and no further viscoplastic flow is possible (the stress

![Figure 3.15: Schematic representation of a creep process according to the over-stress theory: (a) non-hardening material and (b) hardening material](image)

![Figure 3.16: Schematic representation of a stress relaxation process according to the over-stress theory](image)
reaches a constant value).

**Constant strain rate.** Consider now a constant strain rate test from a stress state inside the static yield surface. When the current state of stress is inside the static yield surface ($F < 0$) the viscoplastic strain rate is equal to zero and the elastic strain rate is equal to the total strain rate, according to Equation (3.54). As soon as the current state of stress moves outside the static yield surface, viscoplastic strains will be produced, according to Equation (3.55), and the sum of the viscoplastic and elastic strain rates must now be equal to the imposed strain rate. In this case, the amount of the overstress $F$ must all the time be updated and adjusted in the numerical algorithms in such a way that the total strain rate is constant. In summary, it is possible to model constant strain rate tests.

**Static yield surface versus classical yield surface.** The analogy between the static yield surface of viscoplasticity and the classical yield surface of plasticity has been proposed from various authors (Zienkiewicz et al., 1975; Di Prisco and Imposimato, 1996). They postulated that the integration of the viscoplastic strains over time eventually gives the elastoplastic solution:

$$\int_0^\infty \dot{\varepsilon}^{vp}_{ij} \cdot dt = \Delta \varepsilon^p_{ij}$$

(3.64)

Hashiguchi and Okayasu (2000b) have questioned the fact that overstress theory can be used to obtain classical plastic solutions. They reported that the viscoplastic overstress model is fundamentally different from elastoplasticity. This is due to the fact that viscoplastic straining in the overstress model is not related to the stress rate but to the stress, while plastic straining is related to the stress rate in elastoplasticity.

**Overstress models**

Many viscoplastic models based on the Perzyna’s overstress are found in literature:

- Lemaitre’s model (Boidy, 2000);
- Pragher’s model (Prager, 1949);
- Cristescu’s model (Jin and Cristescu, 1998);
- Bodner and Partom’s model (Bodner and Partom, 1975);
- Adachi and Okano’s model (Adachi and Okano, 1974);
3.4. General theories

- Adachi and Oka’s model (Adachi and Oka, 1982);
- Di Prisco’s model (Di Prisco and Imposimato, 1996);
- Katona’s model (Katona, 1984);
- Zienkiewicz’s model (Zienkiewicz et al., 1975);
- Desai and Zhang’s model (Desai and Zhang, 1987).

In the following section only the Lemaitre’s model is briefly described. This is because this model was widely used within the research group, and because it forms the theoretical fundamental basis of the new model discussed in the following.

**Lemaitre’s model**

This viscoplastic model was proposed by Lemaitre and Chaboche (1990) for metal alloys and extended by Boidy (2000) to the study of the time dependent behaviour of rocks. This model is based on the Perzyna’s overstress theory discussed above.

The main assumption of the Lemaitre’s model is that the static yield surface is reduced to the hydrostatic axis \( q = 0 \) and does not harden by means of the viscoplastic deformations. Therefore, the elastic field is limited to the isotropic states of stress and viscoplastic deformations develop only if the deviatoric state of stress is different from zero.

The yield function \( f \) is assumed to be split into a part \( \bar{f} \), which depends only on the stress state, and into a part \( \kappa \), which depends only on the viscoplastic strains, according to:

\[
 f = \frac{\bar{f}(\sigma_{ij})}{\kappa(\epsilon_{ij}^{vp})} \quad (3.65)
\]

For the function \( \bar{f} \) a Von Mises’s yield criterion is assumed:

\[
 \bar{f} = q \quad (3.66)
\]

where \( q \) is the deviatoric stress:

\[
 q = \sqrt{\frac{3}{2} s_{ij} \cdot s_{ij}} \quad (3.67)
\]

and where \( s_{ij} \) is the stress deviator tensor:

\[
 s_{ij} = \sigma_{ij} - \frac{1}{3} \cdot \delta_{ij} \cdot \sigma_{mm} \quad (3.68)
\]

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A potential hardening rule is introduced by means of the function $\kappa$:

$$\kappa(\varepsilon_{vp}^{ij}) = (\varepsilon_{q}^{vp})^\frac{m}{n}$$

(3.69)

where $m$ and $n$ are constitutive parameters ($n \geq 1; 1 - n < m \leq 0$).

The deviatoric viscoplastic strain $\varepsilon_{vp}^{q}$ is assumed to be:

$$\varepsilon_{q}^{vp} = \int_0^t \dot{\varepsilon}_{q}^{vp} \cdot d\tau$$

(3.70)

where $\dot{\varepsilon}_{q}^{vp}$ is the deviatoric viscoplastic strain rate, that can be evaluated as:

$$\dot{\varepsilon}_{q}^{vp} = \sqrt{\frac{2}{3} \varepsilon_{ij}^{vp} \cdot \varepsilon_{ij}^{vp}}$$

(3.71)

where $\varepsilon_{ij}^{vp}$ is the deviator viscoplastic strain rate tensor:

$$\varepsilon_{ij}^{vp} = \varepsilon_{ij}^{vp} - \frac{1}{3} \delta_{ij} \cdot \varepsilon_{mm}^{vp} = \frac{\partial}{\partial \tau} (\varepsilon_{ij}^{vp})$$

(3.72)

The viscous nucleus $\Phi$ is assumed to be a power law:

$$\Phi(F) = \langle F \rangle^n$$

(3.73)

where $n$ is a constitutive parameter ($n \geq 1$).

The overstress function $F$ is assumed to be represented by the static yield function $f$:

$$F = f$$

(3.74)

The viscoplastic potential function $g$ is supposed to be equal to $\bar{f}$ (i.e. the flow rule is associated):

$$g = \bar{f}$$

(3.75)

With these assumptions, the viscoplastic strains depend only on the deviatoric state of stress and do not induce volumetric strains. Therefore, Equation (3.55) becomes:

$$\dot{\varepsilon}_{ij}^{vp} = \frac{3}{2} \cdot \gamma \cdot q^{n-1} \cdot (\varepsilon_{q}^{vp})^m \cdot s_{ij}$$

(3.76)

The constitutive parameters $n$ and $m$ define respectively the dependence of the viscoplastic strain rate tensor on the deviatoric stress and on the deviatoric viscoplastic strain, whereas the parameter $\gamma$ defines the amplitude of the viscoplastic strain.

Equation (3.76) results in a closed-form solution only when $q$ is constant; otherwise a numerical method is required.
3.4. General theories

Creep. During creep tests the deviatoric stress \( q \) is constant with time and Equation (3.76) can be solved analytically:

\[
\varepsilon_{vp}^q = a \cdot q^\beta \cdot t^\alpha
\]  
(3.77)

where:

\[
\alpha = \frac{1}{1 - m}
\]  
(3.78)

\[
\beta = \frac{n}{1 - m} = n \cdot \alpha
\]  
(3.79)

\[
a = \left(\frac{\gamma}{\alpha}\right)^\alpha
\]  
(3.80)

For a cylindrical specimen subjected to axial and radial stresses \( \sigma_a \) and \( \sigma_r \) it is possible to obtain:

\[
q = \sigma_a - \sigma_r
\]  
(3.81)

\[
\varepsilon_{vp}^q = \varepsilon_{vp}^a = -\frac{1}{2} \cdot \varepsilon_{vp}^r
\]  
(3.82)

Deriving and writing in a logarithmic form Equation (3.77) one obtains:

\[
\log(\dot{\varepsilon}_{vp}^q) = \log(\alpha \cdot a) + \beta \cdot \log(q) + (\alpha - 1) \cdot \log(t)
\]  
(3.83)

From Equation (3.83) it is possible to observe:

- the logarithm of viscoplastic strain rate is a linear function of the logarithm of time;
- the slope of the logarithm of viscoplastic strain rate versus logarithm of time does not depend on the applied stress, as observed by Singh and Mitchell (1968) and Tavenas et al. (1978);
- the logarithm of the viscoplastic strain rate is linear with the logarithm of the deviatoric stress \( q \); this is a good approximation of the relationship proposed by Mitchell (1993);
- Equation (3.83) is very useful to determine the viscoplastic parameters of the model from the results of experimental creep tests (see Bonini et al. (2007), Debernardi (2004)).
3.4.2 Nonstationary Flow Surface theory

The concepts of the Nonstationary Flow Surface (NSFS) theory has been introduced by Naghdi and Murch (1963). This theory is a further extension of the classical inviscid theory of elastoplasticity, i.e. the NSFS theory is based on the basic concepts of theory of elastoplasticity. Therefore, in the following section only the fundamental differences between the theories are presented.

The major difference between the classical theory of elastoplasticity and the NSFS theory lies in the definition of the yield surface. In elastoplasticity the yield function is assumed to be dependent on the effective stress state $\sigma_{ij}$ and on time independent internal variables, as the plastic strain $\varepsilon^p_{ij}$. Consequently it is possible to define the yield surface as:

$$f(\sigma_{ij}, \varepsilon^p_{ij}) = 0 \quad (3.84)$$

According to Equation (3.84) the yield surface does not change with time if the other variables are held constant. In that sense the yield surface can be denoted as “stationary”.

In contrasts with the theory of elastoplasticity, the NSFS surface assumes that the yield function depends directly on time:

$$f(\sigma_{ij}, \varepsilon^{vp}_{ij}, \beta(t)) = 0 \quad (3.85)$$

where $\varepsilon^{vp}_{ij}$ are the viscoplastic strain and $\beta(t)$ a time dependent function. The yield surface can harden with time even if the viscoplastic strains are held constant. In that sense, the flow surface can be denoted as “nonstationary”. The difference between the definition of the yield surface of elastoplasticity and the definition of NSFS is depicted in Figure 3.17.

Two different fields are defined in the effective stress space:

- An elastic field inside the yield surface ($f(\sigma_{ij}) < 0$). Inside this area the strains are only elastic (no viscoplastic deformations develop) and can be determined by using the generalized Hooke’s law of elasticity:

$$\dot{\varepsilon}^e_{ij} = C_{ijkl} \cdot \dot{\sigma}_{kl} \quad (3.86)$$

This field is completely coincident to the elastic field of the elastoplastic theory.

- An elastoviscoplastic field if the point which represents the current effective stress state lies on the yield surface ($f(\sigma_{ij}) = 0$). If a loading condition is considered, elastoviscoplastic strains occurs. Likewise in the overstress theory, the strain rate tensor is assumed to be split into an elastic and a viscoplastic component, to give:
Figure 3.17: Yield surface and stress path for the NSFS theory. For an elastoviscoplastic material, the yield surface can change with time by means of the function \( \beta \), even if the viscoplastic strains are held constant, and can be reached at different points \( A_1, A_2, A_3 \). For an elastoplastic material the yield surface, corresponding to a given plastic strain, does not change with time and is reached at the same point (for example point \( A_1 \)).

\[
\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^{vp} \tag{3.87}
\]

The elastic strain rate is determined by Hooke's generalized law of Equation (3.86).

The viscoplastic strain rate is defined according to a general flow rule, that holds true also for elastoplasticity:

\[
\dot{\varepsilon}_{ij}^{vp} = \langle \Lambda \rangle \cdot \frac{\partial g}{\partial \sigma_{ij}} \tag{3.88}
\]

where \( \Lambda \) is a non-negative multiplier called viscoplastic multiplier and \( g \) is the viscoplastic potential function. The MacCauleys brackets \( \langle \rangle \) ensure that the viscoplastic strains occur when loading from a point on the viscoplastic surface; otherwise, the viscoplastic strains are zero.

The viscoplastic multiplier \( \Lambda \) can be determined by using the consistency condition, which says that viscoplastic loading from a stress state lying on the current yield surface must again lead to a stress state lying on the yield surface (Prager, 1949). The expression for \( \Lambda \) that can be determined as:
Chapter 3. Constitutive models for time dependent behaviour of soils

\[ \Lambda = -\frac{\partial f}{\partial \sigma_{ij}} \cdot \dot{\sigma}_{ij} + \frac{\partial f}{\partial \beta} \cdot \dot{\beta} \]

The multiplier \( \Lambda \) can be viewed as the sum of two contributions \( \Lambda_1 \) and \( \Lambda_2 \), to give:

\[ \Lambda = \Lambda_1 + \Lambda_2 \]  \hspace{1cm} (3.90)

\[ \Lambda_1 = -\frac{\partial f}{\partial \sigma_{ij}} \cdot \dot{\sigma}_{ij} \]  \hspace{1cm} (3.91)

\[ \Lambda_2 = -\frac{\partial f}{\partial \varepsilon_{ijkl}^{vp}} \cdot \frac{\partial g}{\partial \sigma_{ij}} \]  \hspace{1cm} (3.92)

The parameter \( \Lambda_1 \) is identical to the plastic multiplier \( \lambda \) defined in connection with the classical theory of elastoplasticity. Therefore, it can be concluded from Equation (3.90) that the only difference between the plastic multiplier \( \lambda \) and the viscoplastic multiplier \( \Lambda \) is that the latter includes an additional term \((\partial f / \partial \beta) \cdot \dot{\beta}\) in the numerator. This additional term implies that elastoviscoplastic strains occur even though the stress are held constant, which corresponds to a creep process.

The constitutive equations in connection with the NSFS theory can, by using Equations (3.87)-(3.92) and the Hooke’s generalized law, be described as:

\[ \varepsilon = \frac{\dot{\varepsilon}_{ij}}{E} + \left( \frac{\partial f}{\partial \sigma_{ij}} \cdot \dot{\sigma}_{ij} + \frac{\partial f}{\partial \beta} \cdot \dot{\beta} \right) \frac{\partial g}{\partial \sigma_{ij}} \]  \hspace{1cm} (3.93)

According to Naghdi and Murch (1963) and Perzyna (1966) criteria for unloading, neutral loading, and loading can be described as:

\[ f = 0 \quad L(\dot{\sigma}_{ij}, \dot{\beta}) < 0 \quad \text{unloading} \]
\[ f = 0 \quad L(\dot{\sigma}_{ij}, \dot{\beta}) = 0 \quad \text{neutral loading} \]
\[ f = 0 \quad L(\dot{\sigma}_{ij}, \dot{\beta}) > 0 \quad \text{loading} \]  \hspace{1cm} (3.94)
where the operator $L(\dot{\sigma}_{ij}, \dot{\beta})$ is defined by:

$$L(\dot{\sigma}_{ij}, \dot{\beta}) = \frac{\partial f}{\partial \dot{\sigma}_{ij}} \cdot \dot{\sigma}_{ij} + \frac{\partial f}{\partial \dot{\beta}} \cdot \dot{\beta} \quad (3.95)$$

Since time now influences the loading criterion, loading at one rate may be unloading for another, and non-tangent directions to the yield surface may also result in neutral loading. This can be illustrated by use of geometrical interpretation of the loading conditions, which is discussed in the following.

Assume now that the yield condition for an isotropic hardening material in Equation (3.85) can be expressed as:

$$f(\sigma_{ij}, \varepsilon_{ij}^{vp}, \beta) = f(\sigma_{ij}, \varepsilon_{ij}^{vp}) - \kappa(\varepsilon_{ij}^{vp}, \beta) = 0 \quad (3.96)$$

$$\Rightarrow f(\sigma_{ij}, \varepsilon_{ij}^{vp}) = \kappa(\varepsilon_{ij}^{vp}, \beta) \quad (3.97)$$

where $\kappa$ is an hardening function.

Differentiating Equation (3.97) and introducing Equations (3.94) and (3.95) for the neutral loading case it is possible to write:

$$\frac{\partial f}{\partial \sigma_{ij}} \cdot \dot{\sigma}_{ij} - \frac{\partial \kappa}{\partial \beta} \cdot \frac{\partial \beta}{\partial t} = 0 \quad (3.98)$$

$$\Leftrightarrow \cos(\theta) = \left( \frac{\frac{\partial \kappa}{\partial \beta} \cdot \frac{\partial \beta}{\partial t}}{\left| \frac{\partial f}{\partial \sigma_{ij}} \right| \cdot |\dot{\sigma}_{ij}|} \right) \Rightarrow \theta = \arccos \left( \frac{\frac{\partial \kappa}{\partial \beta} \cdot \frac{\partial \beta}{\partial t}}{\left| \frac{\partial f}{\partial \sigma_{ij}} \right| \cdot |\dot{\sigma}_{ij}|} \right) \quad (3.99)$$

where $\theta$ is the plane angle between the stress rate $\dot{\sigma}_{ij}$ and the normal to the yield surface $\partial f / \partial \sigma_{ij}$ in the case of neutral loading. That is, neutral loading can geometrically be symbolized as a cone in stress space with the opening angle $\theta$ as illustrated in Figure 3.18.

The angle between the stress rate vector $\dot{\sigma}_{ij}$ and the normal to the yield surface $\partial f / \partial \sigma_{ij}$ in an arbitrary loading condition (loading, neutral loading, unloading) can be denoted by $\phi$. It can be shown that the complete loading criteria is defined as:

$$\phi > \theta \quad \text{unloading}$$
$$\phi = \theta \quad \text{neutral loading}$$
$$\phi < \theta \quad \text{loading} \quad (3.100)$$

These conditions are illustrated in Figure 3.18. It should be noted that the specification of the physical nature of the time dependent $\beta$ parameter is a key for the Nonstationary Flow Surface.
Observations on the Nonstationary Flow Surface theory

Creep If a creep process initiated at a stress level located inside the viscoplastic yield surface is considered, the NSFS theory will not predict any inelastic strain. Therefore the NSFS theory cannot describe a creep process initiated from a state of stress inside the yield surface.

The NSFS theory is able to predict a creep process only in the case the stress state belongs to the viscoplastic yield surface. Consider a creep process initiated at a point B on the current yield surface \( f \) at a given time \( t \), as illustrated in Figure 3.19. The viscoplastic deformations start to develop because the stress state point B belong to the current yield surface \( f = 0 \). The loading criterion corresponds to a point which coincides with the apex P of the cone describing the loading criterion (Figure 3.18). Viscoplastic strains \( \varepsilon_{ij}^{vp} \) occur according to Equations (3.88) and (3.89), but the term that includes the stress rate \( \dot{\sigma}_{ij} \) disappears.

The development of viscoplastic strains implies that the viscoplastic yield surface expands in the stress space according to Equation (3.85). At a subsequent time \( t_1 \) the imposed constant state of stress B will be inside the new current yield surface \( f_1 \) as illustrated in Figure 3.19. Therefore, because the stress point is inside the current yield surface \( f_1 \), no further viscoplastic deformation should develop during the remaining part of the creep process. But in association with the NSFS theory it is assumed that a if a creep process is initiated at a stress point B on the yield surface, this creep process will continue to occur even though the stress state B at a subsequent time \( t_1 \) will be inside the new yield
3.4. General theories

surface $f_1$.

![Graph showing different creep process](image)

**Figure 3.19:** Different creep process started at a stress state inside (A) or on (B) the viscoplastic yield surface

As an example consider a normally consolidated clay at a stress state $A$ as shown in Figure 3.20. The initial yield surface is $f_1$. The clay is loaded with a stress path from $A$ to $B$. During this stress path only elastic strains occur because the stress state is inside the current yield surface $f_1$. Once point $B$ is reached, the state of stress is held constant over time. Because point $B$ belongs to the yield surface $f_1$ viscoplastic strains start to develop and the creep process starts to develop.

As discussed above, viscoplastic strains develop during a creep process initiated from a stress state lying on the yield surface. Consequently, as assumed by the NSFS theory, the yield surface hardens and expands with time even though the stresses are constant. During the creep process from point $B$ to $C$, the yield surface expands in such way that $f_2$ is the new current yield surface at the end of the process (point $C$).

The clay appears to be over-consolidated although the history of effective stresses is such that the soil is in fact normally consolidated. Therefore, when loading from $C$ to $D$ occurs, the behaviour of the soil is elastic until point $D$ is reached and the response is instantaneous. Moreover, if the clay is loaded from point $D$ to $E$ with a high rate, the stress state is always on the current yield surface and the response is elastoplastic. If the loading rate is quite small the response is elastoviscoplastic. From this example it can be concluded that during a creep process the yield surface expands. This behaviour is consistent with the Bjerrums approach described above.
Relaxation  Consider a relaxation process started from a point within the yield surface. In this case, viscoplastic strains do not develop because the stress state is inside the current yield surface. As the viscoplastic strains are equal to the elastic strains with opposite direction, the effective state of stress remains constant (non relaxation occur). Therefore the NSFS theory cannot describe a relaxation process started inside the current yield surface.

It appears that it is not clear in the literature whether or not the NSFS theory is capable of describing a relaxation process initiated from a point on the yield surface. In principle, it is possible.

Constant strain rate  The NSFS theory is able to describe the behaviour of soil in a range of deformation rates. The slowest rate corresponds to a total strain rate approaching zero. The fastest strain rate corresponds to the rate at which the term which contains time in Equation (3.93) can be neglected. If the term vanishes in Equation (3.93), the constitutive equations for an elastoviscoplastic material are equivalent to the equations for an elastoplastic material.

Nonstationary flow surface models

Many elastoviscoplastic models based on the NSFS theory are found in literature, e.g.:

- Matsui-Abe’s model (Matsui and Abe, 1985)
- Nova’s model (Nova, 1982)
- Sekiguchi’s model (Sekiguchi, 1984)
Other models

In literature it is possible to find out a very wide number of time dependent constitutive models that cannot be included in the present chapter. Only some models, that are particularly significant, are briefly mentioned below.

- A time dependent constitutive model proposed by Kaliakin and Dafalias (1990a,b) for cohesive soils. This model is an improvement of the elasto-plastic bounding surface model and takes into account time dependence with a general flow rule similar to the one proposed by Perzyna. This model is one of the more advanced viscoplastic model, but it is quite complex and its application to real problems is difficult.

- The Borja model, a stress-strain-time relation for cohesive soils in the wet region proposed by Hsieh et al. (1990)

- The endochronic model proposed by Valanis (1971). The prominent feature of this model is that no yield surface is incorporated in the theory. Furthermore, time is included in the equations by means of an intrinsic time scale, which is a material property.

- The subloading surface model extended to elastoviscoplasticity developed by Hashiguchi and Okayasu (2000a).

- A model proposed by Adachi et al. (1987). The model deals with memory and internal variables in the sense that a strain measure is introduced in the constitutive equation instead of real time.

3.6 Conclusions

The main purpose of this chapter is to give a review of the models that have been developed by many authors in order to describe the time dependent behaviour of geomaterials.

As a great number of constitutive models have been proposed, only the theoretical framework of each class of models is discussed. The attempt is to focus only on the main assumptions that have been introduced to describe the time dependent behaviour of geomaterials. The proposed classification distinguishes the models in: (1) empirical models, (2) rheological models, (3) general theories.
It is clear that no model developed so far can handle all of the observed time effects in soils or rocks and can be used to represent all the in situ conditions. Generally, each model favours some aspects of time dependency and it is formulated with reference to a specific in situ problem. Designers should choose the model that matches better the real mechanical time dependent and time independent behaviour of soil/rock of interest for the specific problem to be analysed.

From this review it is possible to state that most constitutive models have been proposed with reference to the time dependent behaviour of clays, while the models proposed for rocks are relatively few. For this reason, and because the behaviour of the rock material of interest in this work is quite similar to the behaviour of hard soils, the research has been extended to all time dependent models and has not been limited to those proposed for weak rocks.

It is also important to take into account the models that have been used successfully in the past to model the excavation of tunnels in squeezing conditions. Therefore the CVISC model, Cividini-Giodas model and Lemaitre’s model, that have been used by our research group, are presented in this section with more emphasis.
Chapter 4

Testing equipment

4.1 Introduction

The present chapter describes the High Pressure Triaxial Apparatus (HPTA) which has been extensively used for testing purpose. Also described is the High Pressure Back Pressure Shear Apparatus (HPBPSA), an innovative apparatus, that has been designed to study the time dependent behaviour of weak rocks and rock joints, see Barla et al. (2007d) and Barla et al. (2008).

4.2 The High Pressure Triaxial Apparatus

The High Pressure Triaxial Apparatus (HPTA) is an innovative equipment that has been designed and built by GDS Instruments Ltd for the of DIPLAB (Disaster Planning Laboratory) Geomechanical laboratory of the Department of Structural and Geotechnical Engineering of Politecnico di Torino. The main purpose of its development is the interest for testing weak rocks at high confining pressure, with the accuracy allowed for by local strain measurements and a fully automatic controlling system.

4.2.1 General description and main features

The High Pressure Triaxial Apparatus (HPTA), which is shown in Figure 4.1, is characterized by a maximum capacity of 250 kN for the vertical load. It can apply a confining pressure up to 64 MPa and an interstitial water pressure up to 32 MPa. The cell can host cylindrical samples with diameter of 50, 70 and 100 mm and height up to 200 mm. It permits to measure with accuracy both the axial and radial local displacements of the sample. The acquisition
Chapter 4. Testing equipment

and the control systems are fully automatic, interconnected to each other, and completely computerized. This allows to strictly control every test condition and to perform tests according to any possible stress path.

Figure 4.1: High Pressure Triaxial Apparatus (a. triaxial cell; b. load frame; c. cell pressure hydraulic actuator; d. back pressure hydraulic actuator; e. acquisition and control system; f. hydraulic tank system)

The apparatus comprises the following components:

- a triaxial cell;
- a load frame;
- two hydraulic actuators;
- a measurement system;
- a data acquisition and control system;
4.2. The High Pressure Triaxial Apparatus

- an hydraulic tank system;
- a lift truck.

The most important characteristics of the HPTA are described in the following.

4.2.2 Triaxial cell

The triaxial cell (Figure 4.2 and 4.3) is the most important element of the whole apparatus: it hosts the specimen and it allows to apply the confinement pressure and the vertical load.

The cell structure is very stiff and massive: it is made by special high strength steel and it can sustain internal pressure up to 64 MPa with negligible deformations.

The triaxial cell is composed by: a basement, a pressure bell, a balancing cell, and a load system.

Basement

The basement is a very massive structure made of single cylindrical block of high strength steel.

On the bottom it is possible to see a small conical housing and five holes for bolts, that permit to fix firmly the whole triaxial cell to the lower plate of the loading frame.

On the upper part of the basement there is a pedestal, fixed by means of four passing bolts. The pedestal is a cylindrical steel element on which the specimen is placed by means of the interposition of a porous stone. It must have exactly the same diameter of the sample (ISRM, 1995).

On the top of the specimen a top cap is placed, by interposing a second porous stone. The top cap has been recently modified and a spherical seat has been introduced, in order to reduce the bending moment that can develop inside the material, because of the non perfect parallelism of the two opposite bases of the specimen (ISRM, 1995).

It is possible to replace the pedestal and the top cap depending on the diameter of the sample (50, 70 or 100 mm).

A neoprene membrane permits to isolate the sample from the pressure chamber containing silicon oil. It is placed around the sample and fixed to the pedestal and to the top cap by using a set of o-rings. Moreover the membrane allows to avoid relative movements between the sample, the top cap, the porous stones and the pedestal, during the installation and instrumentation phases.

Inside the basement, the pedestal and the top cap some conduits are realized in order to accomplish the hydraulic and electrical connections between the
Figure 4.2: Cross section of the triaxial cell
Figure 4.3: Three-dimensional drawing of the triaxial cell
exterior and the interior of the cell. Figure 4.4 shows a schematic representation of these connections.

The main connections are:

- a cell pressure connection, between the hydraulic actuator and the interior of the pressure chamber. This connection is used to pressurize the silicon oil inside the pressure chamber.

- an additional cell pressure connection, between the oil tank and the interior of the pressure chamber. This connection is used to fill and to empty the pressure chamber with oil.

- two back pressure connections, between the hydraulic actuator and the lower face of the sample.

- a pore pressure connection, between the external pressure transducer and the upper face of the sample.

- an extension connection, between the void pump and the upper spherical seat of the top cap.

**Figure 4.4:** Three-dimensional representation of the hydraulic and electrical connections in the basement
4.2. The High Pressure Triaxial Apparatus

- three electrical connections, between the acquisition system and the local displacement transducers (LVDT).

The hydraulic connections are obtained by means of high strength steel pipes with small section (external diameter of 1/8” and thickness of 0.8 mm) in order to reduce the overall deformability of the hydraulic system. The two porous stones permit a uniform distribution of the water pressure in correspondence to the two bases of the specimen.

**Pressure bell**

The pressure bell is a very stiff structure obtained from a single block of high strength steel. It forms, together with the basement, the internal watertight chamber where the silicon oil is injected and pressurized in order to apply the isotropic confinement pressure to the specimen.

The pressure bell is fixed to the basement through a sectioned closure ring, with C section, and a closure crown. The watertightness is obtained by means of a conical coupling with the basement and by means of a plastic o-ring.

At the top of the pressure bell a cylindrical hole permits the passage of the load rod, that transmits the axial load directly to the sample.

**Balancing cell**

The balancing cell is made of high strength steel and it is placed on the top of the pressure chamber. The load rod passes through it and realizes, by means of an annular piston directly connected to the load rod, a pneumatic cylinder with double chambers. The area of the annular piston is exactly equal to the cross area of the load rod. The upper chamber is full of oil and it is connected to the principal pressure chamber of the triaxial cell by means of an external steel pipe. The lower chamber is connected to the exterior by means of a breather. Therefore, the upper chamber always has the same pressure of the principal pressure chamber.

A set of ball bearings permit to reduce the friction between the balancing cell and the load rod. A series of special o-rings make the chambers completely watertight.

Figure 4.5 shows a three-dimensional representation of the balancing cell. The balancing cell essentially accomplishes two fundamental functions.

- The first function is to permit that the measurement of the external load cell, which is attached to the top of the load rod, is equal to the deviatoric load and it is not influenced by the pressure of the oil inside the principal pressure chamber. This is possible because the pressure inside the upper chamber of the balancing cell is equal to the pressure inside the principal
pressure chamber, and because the cross sectional area of the annular piston is equal to the cross sectional area of the load rod. In other words, the force applied by the oil inside the principal chamber to the rod is equal, but opposite in direction, to the force applied by the oil inside the upper chamber of the balancing cell to the piston of the rod.

- The second function is to remove the influence of the vertical movements of the load rod on the pressure of the oil inside the principal chamber. In fact the movement of the road changes the space into the principal chamber that is available for oil. The balancing cell compensates exactly this volume change and avoids the variation of the oil pressure. It is remarked that the stiffness of the oil is very high and only a little change of its volume can cause a very high change of its pressure.

Figure 4.5: Three-dimensional representation of the balancing cell
4.2. The High Pressure Triaxial Apparatus

Load system

The load system is an important part of the triaxial cell, because it transmits the axial load from the load frame to the specimen.

It comprises, as shown in Figure 4.5, a cylindrical rod, an annular piston that forms with the balancing cell a two chamber cylinder, a set of spacers, an internal load cell, and a spherical seat that realizes with the top cap an articulated joint.

It is very important that the friction between the load rod and the triaxial cell is very small, in order to correctly measure the vertical load applied to the sample. In order to avoid the influence of the friction on the measured load an internal load cell has been added at the end of the load rod.

4.2.3 Load frame

The load frame of the HPTA is a very stiff structure that allows to reach the maximum vertical load of 250 kN and the maximum vertical displacement of 100 mm. It is called VIS, which is the acronym of Virtual Infinite Stiffness (the reason of this name will be discussed below). Figure 4.6 shows a schematic drawing of the load frame.

The contrast structure comprises two horizontal beams of high strength steel, with dimensions of 470x250x1000 mm, connected each other by means of two cylindrical uprights, with diameter 100 mm. The vertical position of the upper beam can be changed by using an electrical engine that drives the vertical movement of two secondary uprights directly connected to the beam.

This structure is very stiff and massive, clearly oversized if compared to the maximum allowable load. This makes the deformations of the overall system nearly negligible if compared to the deformations of the specimen. A very stiff structure, like this one, allows to properly evaluate the softening post-peak behaviour of the material (a non-stiff structure will cause the sudden breakage of the sample at peak).

An external load cell, with a full scale of 250 kN, is directly fixed at the bottom of the upper horizontal beam.

The vertical load is applied by means of a sliding plate connected to the lower beam. The movement of the plate is obtained by means of a screw mechanism and a step-by-step electrical engine which is controlled by an internal micro-processor.

The micro-processor can also read the measurement of the load cell. The integration between the load actuator and the load transducer by means of this digital controller has two fundamental advantages. The first one is that it is possible to follow stress paths both under displacement control and under load control. The second one is that the digital controller can correct the vertical
displacement of the plate by subtracting the deformations of the overall contrast structure. This allows to obtain a Virtual Infinite Stiffness system.

The load frame can be controlled directly by using a keyboard pad or remotely by means of a personal computer.

### 4.2.4 Hydraulic actuators

The silicon oil inside the pressure chamber (cell pressure) and the interstitial water on the bottom of the sample (back pressure) are pressurized by means of two digital hydraulic actuators (Figure 4.7). The actuator of the cell pressure can reach the maximum pressure of 64 MPa, while the actuator of the back pressure 32 MPa.

The actuators are formed of a steel cylinder, with a maximum capacity of
4.2. The High Pressure Triaxial Apparatus

Figure 4.7: Hydraulic actuator of the cell pressure (above) and hydraulic actuator of the back pressure (below)

Figure 4.8: Functioning diagram of the hydraulic actuators
200 cm³, and a steel piston that compresses the oil or the water, by means of a step-by-step electrical engine (Figure 4.8). The rotation of the engine is controlled by a digital unit which is also connected to a pressure transducer placed at the end of the cylinder. The presence of this digital unit, that links the actuator to the transducer, allows to perform tests under pressure or volume control.

In a similar manner to the load frame, the hydraulic actuators can be controlled directly, by using a key pad, or remotely, by means of a personal computer.

### 4.2.5 Measurement system

The measurement system is completely digital and comprises: a load measurement system, a pressure measurement system, and a displacement measurement system.

#### Load measurement system

The load measurement system allows to measure the deviatoric load applied to the sample. It comprises an external load cell, with a full scale of 250 kN and an accuracy of 60 N, placed between the upper horizontal beam of the load frame and the upper plate of the triaxial cell, and an internal load cell fixed at the bottom of the load rod inside the triaxial cell. The apparatus is equipped with two different internal load cells (Figure 4.9): the first one has a full scale of 64 kN and an accuracy of 15 N, while the second one has a full scale of 250 kN and an accuracy of 60 N. It is possible to use one of these load cells.
4.2. The High Pressure Triaxial Apparatus

cells depending on the applied load and on the accuracy required. The presence of an internal load cell permits to have an exact measurement of the deviatoric load applied to the sample, not influenced by the friction between the load rod and the balancing cell.

**Pressure measurement system**

The pressure measurement system is partially integrated into the hydraulic actuators. The measurement of the pressure of the silicon oil inside the pressure chamber (cell pressure) and the pressure of the interstitial water at the bottom of the sample (back pressure) is done at actuator level, with an accuracy of 16 kPa and 8 kPa, respectively.

The system is completed by an external pressure transducer, that allows to measure the pressure of the interstitial water at the top of the sample (pore pressure). This transducer has a full scale of 32 MPa and an accuracy of 8 kPa. The possibility to measure separately the pressure of the interstitial water at the bottom (back pressure) and at the top (pore pressure) of the sample, permits to perform permeability tests.

**Displacement measurement system**

The displacement measurement system is subdivided into an external and local displacement measurement system.

The external system allows to evaluate the vertical displacement of the sliding plate of the load frame, by means of the rotation angle of the raising screw mechanism. This system is integrated into the digital unit of the load frame and is characterized by an accuracy of 1 µm. It is called “external” because it includes, in addition to the real deformation of the specimen, also the deformation of the triaxial cell and the possible gaps between the parts. It is important to remember that it does not include the deformation of the contrast structure, because they are automatically subtracted by the internal digital unit. For stiff materials the external measurement can differ considerably from the real displacement of the specimen.

In order to measure with more accuracy the real deformation of the specimen the apparatus is equipped with a local displacement measurement system. This system comprises a set of LVDT’s (Linear Variable Differential Transformer) that can be placed directly on the lateral surface of the sample. They are usually glued on the membrane. This system is very accurate and it is composed of two axial transducers, which are diametrically opposed, in order to compensate a possible bending moment inside the sample, and by a radial transducer. Figure 4.10 shows a sample instrumented with the local transducers. The
accuracy of the vertical measurements is 1 µm on a full scale of 10 mm, while the accuracy of the radial measurement is 0.5 µm on a full scale of 5 mm.

4.2.6 Data acquisition and control system

The data acquisition and control system are organized on two hierarchical levels, as shown in Figure 4.11.

At lower level, the control systems are connected to the respective measurement systems by means of a digital unit, in order to create some self-governing components (e.g. the back pressure transducer is connected to the stepping engine that pressurizes the water by means of the digital unit inside the hydraulic actuator).

At higher level, these components are connected, with the remaining measurement systems, to a personal computer that controls the whole apparatus by
4.2. The High Pressure Triaxial Apparatus

This hierarchical organization allows the user to control the apparatus both locally, through the key pads of the components, and remotely, through the software installed on the personal computer.

The digital integration of the measurement system with the control systems permits to control automatically every test parameters and boundary conditions. It makes it possible to perform both drained and undrained tests, according to possible stress path, tests with constant strain rate or constant load rate, tests with radial displacement equal to zero ($K_0$ tests), tests with constant radial strain rate, creep tests, relaxation tests, permeability tests, and so on.

### 4.2.7 Hydraulic tank system

The apparatus is equipped with two different hydraulic tank systems (Figure 4.12): one for the silicon oil inside the pressure chamber, and one for the interstitial water inside the sample.
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Figure 4.12: Hydraulic tank system (a. oil tank; b. water tank; c. control panel; d. vacuum pump; e. water lung)

Oil tank system

The oil tank is connected to the pressure chamber of the triaxial cell, by means of a large diameter pipe into the basement, and to the hydraulic actuator of the cell pressure. It can be pressurized or depressurized through a control panel, which is connected to the air pressure circuit and to the void pump.

The system allows to perform the following operations:

- Deaerate the oil. This is a very important procedure, because the air, which is dissolved into the oil, can reduce considerably its stiffness, with bad consequences during the pressurization phases.
4.2. The High Pressure Triaxial Apparatus

- Fill or empty the pressure chamber. By applying a small air pressure to the oil tank or to the pressure chamber it is possible to fill or empty the pressure chamber of the triaxial cell. A large diameter pipe has been introduced into the basement of the cell in order to minimize the filling or emptying time.

- Fill the hydraulic actuator of the cell pressure. In order to avoid the cavitation of oil, this procedure must be done with a small air pressure applied to the oil tank.

**Water tank system**

The water tank system is composed of a perspex tank and a perspex “lung”, connected, by means of the control panel, to the air pressure circuit and to the void pump.

The water tank allows to deaerate the distilled water and permits to fill the lung and the hydraulic actuator of the back pressure.

The lung is equipped with a rubber membrane, that avoids the direct contact between water and air, in order to prevent the dissolution of air into water. It permits to apply a constant water pressure on the bottom of the sample, in order to obtain a slight water flow inside the material. This procedure is used to saturate all the pores of the sample and it is called flushing procedure (Barla, 1999).

**4.2.8 Lift truck**

In order to raise and move the triaxial cell, that weights approximately 350 kg, in a simple and secure manner, a new lift truck has been specifically designed and built. Figure 4.13 shows this lift truck in action.

The lift truck is a self-moving structure, that has been designed exactly on the dimensions of the High Pressure Triaxial Apparatus. It is equipped with a tackle driven by an inverter engine, that allows to raise up and down the triaxial cell, through a couple of steel-wire ropes and pulleys. The tackle is fixed to an horizontal trolley, that permits to translate the cell horizontally, through a manual handle. Moreover the truck is equipped with a working table, on which it is possible to lean the basement of the triaxial cell.

This new lift truck makes it easy the installation of the triaxial cell on the load frame, the operations of opening and closure of the cell and all the other operations that need to be made in order to move the cell. All these operations can be done easily and securely by only one person.
4.3 High Pressure Back Pressure Shear Apparatus

The High Pressure Back Pressure Shear Apparatus (HPBPSA) is a servo controlled direct shear apparatus that has been recently designed and developed by Politecnico di Torino and G.D.S. Instruments Ltd (Figure 4.14).

The apparatus allows tests to be performed under closely controlled conditions in terms of axial load, shear load, and back pressure applied to the specimen. In addition, tests can be performed on rock joints/interfaces with fluid pressure imposed.

The system comprises a shear box which is inserted into a pressure chamber. Loading actuators and a pressure controller allow the operator to apply appropriate loading conditions to the specimen. A measuring and acquisition system is directly connected to a control unit based on a personal computer. A
4.3. High Pressure Back Pressure Shear Apparatus

Figure 4.14: High Pressure Back Pressure Shear Apparatus (a. pressure chamber; b. vertical load system; c. horizontal load system; d. back pressure hydraulic actuator; e. acquisition and control system; f. water tank system)

A schematic view of the apparatus is shown in Figure 4.14.

Figure 4.15 shows a sketch of the shear box placed inside the pressure chamber. It consists of a lower and an upper part. The upper part of the box is rigidly connected to the top of the pressure chamber by four bolts. Shearing of the specimen is achieved by moving the lower part of the box in the horizontal direction. The shear box can host specimens of 50 or 100 mm in diameter. If a test is to be conducted on a joint, different gaps (up to a maximum of 10 mm) may be set between the two parts of the shear box, due to the presence or absence of a metal plate with a given thickness located below it.

The vertical load is applied to the specimen by a transverse steel bar joined both to the loading ram and to two uprights. A servo-controlled electrical motor allows the transverse bar to shift downward applying load to the specimen. The horizontal load also is applied by a servo-controlled loading ram connected to
Figure 4.15: Cross section of the shear box and the pressure chamber

the pedestal of the shear box. The maximum loading capacity both in the axial and the horizontal direction is 100 kN.

The back pressure is applied by an hydraulic servo-controlled actuator completely similar to the one of the HPTA (see 4.2.4). The maximum pressure that can be achieved is 10 MPa.

A water tank is available for filling both the pressure chamber and the hydraulic actuator. The tank is connected to a vacuum pump so that water can be deaerated before being introduced into drainage circuit.

The apparatus is equipped with a number of measuring transducers that allow for testing to be performed in closely controlled conditions.

The axial load is measured by a load cell integrated into the loading ram, with a full scale of 100 kN and an accuracy of 24 N. The shear load is measured by a special load cell connected to the pedestal of the shear box and to the horizontal loading ram. The full scale of the shear load cell is 100 kN and the accuracy is 24 N. To prevent the cell from any damage, it is submerged in silicon
oil, separated from the water in the pressure chamber by a rubber membrane.

The pore pressure is measured by a pressure transducer in the hydraulic actuator.

The vertical displacement is measured at two external location. The first measurement is given by the rotation of the driving system of the electrical motor that moves the loading ram (accuracy of 1 µm). The second measurement is given by a potentiometer transducer directly connected to the loading ram and positioned over the pressure chamber (full scale of 50 mm and accuracy of 12 µm).

The horizontal displacement are monitored at three different position. Again, two external measurements are due to the loading system (accuracy of 1 µm) and to a potentiometer transducer directly connected to the shear loading ram (full scale of 50 mm and accuracy of 12 µm). An additional shear displacement measurement is taken inside the pressure chamber by means of an LVDT assembly (full scale of 5 mm and accuracy of 1 µm). The LVDT is rigidly connected to the lower and the upper parts of the shear box. By measuring the displacement at this location, the deformability of the loading ram is not affecting the measurement.

The acquisition and control system is similar to the one of HPTA (see 4.2.6) and it is structured into two hierarchical levels. At lower level it is possible to control the vertical load, the horizontal load and the back pressure locally, through the key pads of the systems, or remotely, using the personal computer. At higher level a digital unit connected to the personal computer allows to control all the load actuator and to measure all the transducers, through a specific software (GDS Lab).
Chapter 4. Testing equipment
Chapter 5

Laboratory testing of coal

5.1 Introduction

The availability of rock samples obtained from the Saint Martin La Porte access adit along the Torino-Lyon Base Tunnel, which experienced very important squeezing deformations during excavation, and the peculiar characteristics of the material has determined the choice of coal as the rock material of interest for the present study (Barla et al., 2007b).

An experimental program has been performed on coal, involving the determination of the physical properties and mineralogical composition, in conjunction with oedometer tests, direct shear tests and triaxial tests, performed in closely controlled conditions by using the two laboratory equipments described in Chapter 4.

The aim is to evaluate the most important aspects of the mechanical behaviour of coal, which can influence the tunnel response, with particular attention to its time dependent behaviour.

5.2 The site

5.2.1 General description

The Saint Martin La Porte access adit (cross section of 80 m² and length of 2050 m) is one of the three access adits, on the French side, of the Torino-Lyon Base Tunnel (Figure 5.1).

It comprises an initial climbing part with slope of 1.1% and curvature radius of 710 m, a subsequent rectilinear descending segment with slope of 7.8%, and a final segment with slight slope and main axis rotated rightwards of 40°, as
shown in Figure 5.2. The total difference in height is 84 m, from the entrance to the connection point to the base tunnel, where the maximum overburden is 600 m.

As shown in Figure 5.3, where the geological profile is illustrated, a cone of detritus is present near the tunnel portal up to chainage 60 m. Then the rock mass is composed of limestone, marls, and dolomitic rock. From chainage 600 to 800 m, the Carboniferous Formation, which is composed principally of gypsum and anhydrite, is encountered. From chainage 800 m the tunnel enters into the Productive Carboniferous Formation, which is composed of sandstone, schists, and veins of coal.

From chainage 1280 to 1400 m (Figure 5.3) the tunnel underwent severe squeezing deformations, which caused some serious problems, as: large convergences (up to 2 m), local failures, local instabilities, and re-profiling necessity. An advanced monitoring system has been installed in order to quantify exactly the tunnel response during excavation. It makes this section of the tunnel an excellent case of study to investigate the viscoplastic behaviour of weak rocks in relation with tunnel design.

The considered part of the tunnel is characterized by an overburden from 250 m to 400 m. No evidences of the presence of water table have been found during excavation. The state of stress prior to excavation is unknown because no direct measurement was performed. It is being excavated in the Carboniferous Formation - “Zone Houillère Briençonnaise-Unité des Encombes” (hSG in Figure 5.3), which is composed of black schists (45 to 55 %), sandstone (40 to 50 %), coal (5 %), clay-like shales and cataclastic rocks.
5.2. The site

**Figure 5.2:** Plan of the Saint Martin La Porte access tunnel (Rettighieri et al., 2008)

**Figure 5.3:** Geological profile along the Saint Martin La Porte access tunnel (Rettighieri et al., 2008)
Chapter 5. Laboratory testing of coal

Figure 5.4: Typical geological conditions at the face (a. grey clay shales; b. black schists; c. coal; d. cataclastic rocks; e. grey schists; g. sandstone; gps. psammitic-sandstone; hsg. houiller (schists-sandstone); stkw. stokwerk; q. quartz inclusion)

A characteristic feature of the ground as observed at the face during excavation (Figure 5.4) is the highly heterogeneous, disrupted and fractured condition of rock mass. The formation is often affected by faulting which results in a degradation of the rock mass conditions.

The main assumption, that has been advanced in this study, is that the time dependent behaviour of the rock mass, that generates the squeezing phenomenon, is due only to the “weakest” part of the rock mass (coal, clay-like shales, and cataclastic rocks) and to the fracturing system. Therefore, it has been decided to focalize the experimental research on the evaluation of the geomechanical characteristics of coal, with particular attention to its strength and time dependent behaviour. This decision has been made essentially for two reasons: (1) the coal is the weakest rock, which is present in the greatest quantity; (2) it has been possible to obtain cylindrical specimens only from the blocks of coal (see Section 5.3), but not from the blocks of clay-like shale and cataclastic rock, because they are too brittle, fractured and disrupted.
5.2.2 Taking the material

It has not been possible to take intact blocks of coal directly from the tunnel face, because of the excavation method (pick hammer) and the high brittleness of the material.

For this reason it has been decided to use core samples (Figure 5.5), obtained from a neighbouring borehole (F133), at the same depth of the considered section of the tunnel. Table 5.1 gives the main properties of the samples.

![Figure 5.5: Typical core sample of coal retrieved from borehole](image)

<table>
<thead>
<tr>
<th>Sample</th>
<th>Box</th>
<th>Depth from [m]</th>
<th>Depth to [m]</th>
<th>Length [mm]</th>
<th>Diameter [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A10</td>
<td>178</td>
<td>394.20</td>
<td>394.40</td>
<td>200</td>
<td>85</td>
</tr>
<tr>
<td>A11</td>
<td>179</td>
<td>397.40</td>
<td>397.70</td>
<td>300</td>
<td>65</td>
</tr>
<tr>
<td>A12</td>
<td></td>
<td>398.50</td>
<td>399.00</td>
<td>500</td>
<td>65</td>
</tr>
<tr>
<td>A13</td>
<td></td>
<td>399.50</td>
<td>400.00</td>
<td>500</td>
<td>65</td>
</tr>
<tr>
<td>A15</td>
<td>208</td>
<td>507.40</td>
<td>507.65</td>
<td>250</td>
<td>65</td>
</tr>
<tr>
<td>A16</td>
<td>208</td>
<td>508.40</td>
<td>508.80</td>
<td>400</td>
<td>65</td>
</tr>
<tr>
<td>A17</td>
<td>208</td>
<td>509.50</td>
<td>510.20</td>
<td>700</td>
<td>65</td>
</tr>
</tbody>
</table>
Once in laboratory, each sample have been wrapped into a plastic foil, inserted into a PVC tube, and then sealed with paraffin. The samples has been kept in a constant temperature and humidity room before testing, in order to avoid desiccation.

### 5.3 Specimen preparation

Specimen preparation needed to be carried out with great care in order to avoid any possible disturbance. It has been performed with great difficulties because of the high brittleness of the material, and the presence of inclusions (quartz and calcite), and weakness planes. Samples have been taken to the laboratory, where cylindrical specimens have been cut by using a metal lathe. Particular attention has been posed on the preparation of the two bases in order to obtain good planar and satisfactory parallel surfaces. Table 5.2 gives the dimensions and the weight of the 6 specimens finally obtained.

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Block</th>
<th>Diameter [mm]</th>
<th>Height [mm]</th>
<th>Weight [g]</th>
<th>Test performed</th>
</tr>
</thead>
<tbody>
<tr>
<td>A11a</td>
<td>A11</td>
<td>50.0</td>
<td>100.1</td>
<td>348</td>
<td>Triaxial strength</td>
</tr>
<tr>
<td>A17a</td>
<td>A17</td>
<td>50.1</td>
<td>100.0</td>
<td>347</td>
<td>Triaxial creep</td>
</tr>
<tr>
<td>A17b</td>
<td>A17</td>
<td>50.1</td>
<td>100.3</td>
<td>347</td>
<td>Triaxial creep</td>
</tr>
<tr>
<td>A17c</td>
<td>A17</td>
<td>50.0</td>
<td>54.5</td>
<td>189</td>
<td>Oedometer-shear</td>
</tr>
<tr>
<td>A17d</td>
<td>A17</td>
<td>50.0</td>
<td>100.0</td>
<td>350</td>
<td>Triaxial creep</td>
</tr>
<tr>
<td>A17e</td>
<td>A17</td>
<td>50.0</td>
<td>50.84</td>
<td>—</td>
<td>Shear</td>
</tr>
</tbody>
</table>

Each specimen has been wrapped into a plastic foil in order to avoid any contact with air. The protecting layer has been removed only at the time of inserting the specimen in the triaxial cell for testing.

### 5.4 Physical properties and mineralogical composition

The mean physical properties of coal are given in Table 5.3. It is possible to observe that the value of porosity is rather high and is more similar to the porosity of a soil than a rock.

Although it is not possible to consider coal as a classical geotechnical material, like clay, the index properties of the material (Atterberg limits) have been determined. The coal samples have been in advance broken into little
Table 5.3: Physical and index properties of coal (mean values)

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volumetric weight</td>
<td>$\gamma$</td>
<td>17.29 kN/m$^3$</td>
</tr>
<tr>
<td>Specific grain weight</td>
<td>$G_s$</td>
<td>1.99</td>
</tr>
<tr>
<td>Porosity</td>
<td>$n$</td>
<td>15.94 %</td>
</tr>
<tr>
<td>Void index</td>
<td>$e$</td>
<td>0.19</td>
</tr>
<tr>
<td>Water content</td>
<td>$w_n$</td>
<td>3.36 %</td>
</tr>
<tr>
<td>Saturation</td>
<td>$S$</td>
<td>35.26 %</td>
</tr>
<tr>
<td>Liquid limit</td>
<td>$w_L$</td>
<td>36.21 %</td>
</tr>
<tr>
<td>Plastic limit</td>
<td>$w_P$</td>
<td>25.71 %</td>
</tr>
<tr>
<td>Plastic index</td>
<td>$IP$</td>
<td>10.56 %</td>
</tr>
<tr>
<td>CaCO$_3$ content</td>
<td></td>
<td>0.70 %</td>
</tr>
</tbody>
</table>

bits by means of a soft rubber pestle, and then submerged in water for few days, in order to destroy the structure and achieve a granular material. This material has been then treated like a classical soil in order to determine the Atterberg limits. The obtained values for the liquid and plastic limits are given in Table 5.3. According to the Plasticity Chart of Figure 5.6, coal can be classified as “average compressibility inorganic silt or organic silt”.

Figure 5.6: Plasticity chart for coal

A grain size distribution has been determined by using the same preparation procedure of the material described above (Figure 5.7). Coal is essentially
composed of sand and silt. A small percentage of clay particles (7%) is noted.

![Grain size distribution of coal](image)

**Figure 5.7:** Grain size distribution of coal

The mineralogical composition of coal has been evaluated by using a series of X-ray diffraction analysis. The results obtained are summarised in Table 5.4.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Quartz [%]</th>
<th>Mica [%]</th>
<th>Chlorite [%]</th>
<th>Carbon [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A11</td>
<td>20</td>
<td>15</td>
<td>20</td>
<td>45</td>
</tr>
<tr>
<td>A13</td>
<td>15</td>
<td>15</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>A17</td>
<td>30</td>
<td>5</td>
<td>15</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 5.4: Mineralogical composition of coal from X-ray diffraction analysis

1 amorphous or crystalline phases of carbon

Coal is essentially composed of quartz, mica (muscovite/illite), chlorite (clino-oloro) and amorphous or crystalline phases of carbon. The graphite analysis is not possible because its peaks overlap that of quartz.

The treatment with ethylene glycol has not shown the presence of expansive clay minerals, like smectite (montmorillonite).
5.5 Ultrasonic tests

Some ultrasonic tests have been performed on coal specimens in order to evaluate the sonic velocity of the material (Figure 5.8). The measured travelling times for some specimens are given in Table 5.5, in conjunction with the sonic velocity, density, and dynamic elastic modulus.

![Ultrasonic test apparatus](image)

**Figure 5.8: Ultrasonic test apparatus**

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Travelling time [µs]</th>
<th>Height [mm]</th>
<th>Sonic velocity [m/s]</th>
<th>Density [kg/m³]</th>
<th>Elastic modulus [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A11a</td>
<td>100</td>
<td>100.1</td>
<td>1002</td>
<td>1767</td>
<td>1478</td>
</tr>
<tr>
<td>A17a</td>
<td>93</td>
<td>100.0</td>
<td>1076</td>
<td>1761</td>
<td>1699</td>
</tr>
<tr>
<td>A17b</td>
<td>80</td>
<td>100.3</td>
<td>1254</td>
<td>1756</td>
<td>2302</td>
</tr>
<tr>
<td>A17c</td>
<td>56</td>
<td>54.5</td>
<td>974</td>
<td>1771</td>
<td>1401</td>
</tr>
</tbody>
</table>
5.6 Free deformation tests

The coal samples have been obtained from a borehole at great depth, kept for few months inside wooden boxes in a storehouse, and then taken to the laboratory, where cylindrical specimens have been cut with great difficulty. Therefore the current state of stress, and, probably, the water content, are different from the in situ conditions. The specimens cannot be considered undisturbed.

In order to evaluate if a deferred deformation process is taking place, a free deformation test has been performed on specimen A17a. This test is very simple and consists in measuring the deformations of the sample unloaded versus time. For simplicity in this test only the axial deformation has been measured (Figure 5.9).

![Free deformation test equipment](image)

**Figure 5.9:** Free deformation test equipment

Figure 5.10 shows the axial strain versus time diagram. It is possible to observe that the sample extends at an approximately constant rate.
5.7 Oedometer and direct shear tests

Simultaneously to this test, the variation of the water content into a sample of coal, taken from the same block of specimen A17a, and placed in the same room, has been measured over time. The obtained curve is given in Figure 5.10. It is possible to notice that the trend of the deferred axial strain is very similar to the trend of the variation of water content. Therefore it is possible to state that the observed time dependent deformations of the specimens are essentially due to the variation of water content, and are not related to a viscous effect connected to unloading.

5.7 Oedometer and direct shear tests

In order to evaluate the volumetric and shear behaviour of coal, two tests have been performed by means of the High Pressure Back Pressure Shear Apparatus (HPBPSA) described in Section 4.3. The first test (A17c) has been carried out to study the mechanical time dependent behaviour of coal under oedometer and direct shear conditions. The second test (A17e) has been carried out to study the peak and residual shear strength characteristics of the material. Both tests have been performed in dried conditions, without water, in order to correctly represent the in-situ conditions.
5.7.1 Test A17c

This test can be subdivided into two different parts: (1) an oedometer compression phase, and (2) a direct shear phase. Particular attention has been posed on the time dependent behaviour of the material.

Oedometer compression phase

During this phase the axial loading has been increased gradually from 0 to 50 MPa with a constant rate of 0.05 MPa/min, while the horizontal displacement has been kept constant and equal to zero. The HPBPSA has been used as an high pressure oedometer. The loading phase has been interrupted at different stress levels in order to measure the time dependent deformations of the sample versus time (creep tests). Table 5.6 gives the main results of this phase. For the axial stress of 15 MPa an unloading-reloading phase of 5 MPa has been performed in order to evaluate the material elastic stiffness.

<table>
<thead>
<tr>
<th>Stage</th>
<th>Axial stress [MPa]</th>
<th>Axial strain [%]</th>
<th>Creep duration [days]</th>
<th>Creep strains [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.468</td>
<td>8.9</td>
<td>-0.024</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.655</td>
<td>6.8</td>
<td>0.020</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>0.967</td>
<td>8.0</td>
<td>0.006</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>1.235</td>
<td>10.7</td>
<td>0.009</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>1.479</td>
<td>16.9</td>
<td>0.000</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>1.352</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>1.490</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td>1.651</td>
<td>13.1</td>
<td>-0.011</td>
</tr>
<tr>
<td>9</td>
<td>30</td>
<td>1.961</td>
<td>17.9</td>
<td>0.029</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
<td>2.237</td>
<td>15.0</td>
<td>0.015</td>
</tr>
<tr>
<td>11</td>
<td>50</td>
<td>2.499</td>
<td>18.0</td>
<td>0.030</td>
</tr>
</tbody>
</table>

1-referred to the start of test; 2-referred to the start of creep phase

Figure 5.11 illustrates the obtained stress-strain curve in the semi-logarithmic diagram, which is commonly used in soil mechanics to represent the results of an oedometer test. It is possible to observe that the stress-strain relationship is strongly non linear over the entire stress range. It is not possible to identify the preconsolidation pressure using the Casagrande procedure.

Figure 5.12 shows the creep curves that have been measured for different stress levels. It is possible to observe that the time dependent strains are quite small and similar to each other. Moreover, it is possible to observe that, for some stress levels, the creep curves tend to decrease after an initial increment.
5.7. Oedometer and direct shear tests

**Figure 5.11:** Strain-stress curve for the oedometric compression phase of test A17c

**Figure 5.12:** Creep curves for the oedometer compression phase of test A17c
This is probably due to the superimposition of the phenomenon observed in 5.6, which is caused by the decrease of the water content inside the material.

Although the creep deformations are rather small, it is very interesting to observe the behaviour of the material when the specimen is reloaded after a creep phase, as depicted in Figure 5.13.a. The stress-strain curve is characterised by a higher stiffness than that of the loading phase prior to creep. This stiffness is nearly equal to the stiffness of the unloading phase (Figure 5.13.b), to mean that in the loading phase after creep the deformations are nearly elastic. Further increments of the axial load bring again the point on the original loading curve and the stiffness decreases to the value prior to creep. In other words, when the original curve is reached, inelastic deformations start to develop. In literature, this behaviour is called apparent preconsolidation effect (Section 2.6.1).

**Figure 5.13:** Apparent preconsolidation effect (ageing) after a creep phase

**Direct shear phase**

After the oedometer compression phase a direct shear phase has been performed in order to evaluate the shear behaviour and the strength characteristics of coal. During this phase the vertical stress has been maintained constant and equal to 50 MPa, while the horizontal stress has been increased with a constant rate of displacement equal to 0.01 mm/min. A peak has been reached for a shear stress of 30.86 MPa and a shear strain of 6.28%. Figure 5.14 shows the stress-strain
5.7. Oedometer and direct shear tests

Figure 5.14: Stress-strain curve for the direct shear phase of test A17c

Figure 5.15: Creep curve for the direct shear phase of test A17c
curve obtained. It is possible to observe a large softening phase after reaching peak.

The shear loading phase has been interrupted for a shear stress of 10 MPa in order to perform a shear creep test. Both the vertical load and the horizontal load have been kept constant while the horizontal displacement has been measured over time. The measured creep curve is shown in Figure 5.15.

### 5.7.2 Test A17e

This test has been carried out with the intent to evaluate the peak and residual shear strength of coal. It can be conceptually subdivided in two parts: (1) a multi-stage direct shear phase, and (2) a cyclic direct shear phase.

#### Multi-stage direct shear test

The multi-stage direct shear test has been carried out with the aim to determine more than one peak shear strength point on the same sample.

After the setting-up procedure, the vertical stress has been increased up to 5 MPa with a constant rate of 0.16 MPa/min, while the horizontal displacement has been kept constant and equal to zero. The obtained stress-strain curve is illustrated in Figure 5.16 into a semi-logarithmic diagram. By comparing this curve with the oedometer compression curve of test A17c, it is observed that a very similar behaviour occurs with a little higher compressibility in the first case.

After this phase the horizontal stress has been increased gradually with a constant rate of displacement of 0.005 mm/min, while the vertical stress has been kept constant and equal to 5 MPa, until a peak on the stress-strain diagram has been reached (Figure 5.17). The peak shear stress value is given in Table 5.7.

<table>
<thead>
<tr>
<th>Vertical stress [MPa]</th>
<th>Horizontal stress [MPa]</th>
<th>Horizontal strain [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.00</td>
<td>9.48</td>
<td>2.79</td>
</tr>
<tr>
<td>15.00</td>
<td>16.08</td>
<td>4.64</td>
</tr>
</tbody>
</table>

*1 referred to the start of test*

Immediately after reaching the peak value, the vertical stress has been increased up to 15 MPa, with a constant rate of 0.5 MPa/min while the horizontal displacement has been kept constant.
Then the horizontal stress has been increased with a constant rate of displacement of 0.05 mm/min, while the vertical stress has been kept constant and equal to 15 MPa, until a second peak has been reached on the stress-strain curve (Figure 5.17). The obtained shear stress is given in Table 5.7.

Immediately after reaching the second peak, a complete and unexpected failure did develop inside the specimen. Therefore it has not been possible to evaluate the shear strength at higher vertical stress levels.

Figure 5.17 shows the stress-strain curves for the two shear phases, and highlights the two peak points. If the stress-strain shear curve of this test is compared to the one obtained for the shear phase of test A17c, it is possible to observe approximately the same slope, with the exception of the first part of the curve of test A17c, where the material is apparently less stiff.

**Cyclic direct shear test**

After failure, the axial stress has been increased up to 25 MPa and a cyclic shear test has been performed in order to evaluate the residual shear strength of the material. The vertical stress has been held constant and equal to 25 MPa, while the horizontal displacement has been varied according to a sinusoidal function, defined by an amplitude of ±15 mm and a period of 120 min. 10 cycles have been performed. The stress-displacement diagram is illustrated in Figure 5.18. The residual shear stress is equal to 10.2 MPa.
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Figure 5.17: Stress-strain curve for the direct shear phase of test A17e

Figure 5.18: Stress-displacement diagram for the cyclic direct shear phase of test A17e
5.8 Triaxial tests

A total of four tests have been carried out on coal in order to characterize its strength and time dependent behaviour. The aim has been to simulate at laboratory scale the tunnel behaviour in the short and long term conditions. All the tests have been performed in dried conditions. The experimental equipment used for testing is the High Pressure Triaxial Apparatus (HPTA) described in Section 4.2.

5.8.1 Setting up procedure

Preliminary tests performed on artificial samples pointed out some serious problems caused by punching of the membrane under high confining pressure. This problem occurred also if a special membrane, made of neoprene and 1 mm thick, was used. It was observed that holes developed in correspondence of small defects or pores in the specimens, at the base of the local transducers, and at the contact between the specimen and the pedestal, and between the specimen and the top cap.

Therefore a special setting up procedure needed to be adopted as follows:

(1) The most important defects in the sample are repaired with a special mixture of plaster and coal dust, having nearly the same consistency and stiffness of coal (Figure 5.19.b).

(2) The lateral surface of the specimen is prepared with a special liquid latex (primer), which is used in the building industry to treat walls before painting (Figure 5.19.c). The latex is spread with a soft brush. It allows the next treatment with liquid latex to stick better on the surface of the sample. Moreover, it makes the surface waterproof and avoids the absorption in depth of the liquid phase of the following treatments.

(3) A dense liquid latex, which is commonly used to prepare stamps, is spread with a soft brush on the lateral surface of the specimen (Figure 5.19.d). Few coats are needed. The latex stops all the little pores and defects at the surface and makes it rubbery.

(4) A special water sealing is spread on the surface of the specimen in order to repair the remaining defects, to smooth the surface and make it soft (Figure 5.19.e).

After this treatment, that can takes 1 or 2 days, a neoprene membrane (1 mm thick) is placed around the specimen, and the specimen is placed on the basement of the triaxial cell. Then, few coats of dense liquid latex are spread
Figure 5.19: Setting-up procedure of the coal specimens for the triaxial tests

with a soft brush at the interfaces between the specimen and the pedestal and between the specimen and the top cap. Moreover a thin layer of water sealing is smeared over the latex at the two interfaces, in order to fill and smooth all the imperfections (Figure 5.19.f).

After this, the membrane is turned down and fixed to the pedestal and to the top cap with a set of o-rings. Then, the local transducers are glued on the
5.8. Triaxial tests

membrane. It is quite important to use an elastic glue (like shoe mastic) and not a stiff glue (like cyanoacrylate adhesive or epoxy resin), in order to allow the membrane to deform without creating holes or rips.

5.8.2 Test A11a

This test has been performed with the intent to determine the stiffness and strength properties of coal. It is a multi-stage strength test (ISRM, 1995) and permits to evaluate more than one point on the strength envelope of the material by using only one specimen. It is quite similar to the multi-stage shear strength phase of test A17e (see Section 5.7.2).

Four compression stages have been carried out at different confining pressures of 5, 7.5, 10 and 12.5 MPa, by applying a constant axial displacement rate of 0.01 mm/min, until a peak has been reached on the stress-strain curve. At peak, the confining pressure has been increased while the deviatoric stress has been kept constant in order to allow for the determination of a new peak strength. This process has been repeated four times allowing one to determine four points on the failure envelope as shown in the $t - s$ diagram of Figure 5.20. Immediately after the fourth peak a complete rupture has developed into the sample; therefore it has not been possible to determine further strength points at higher confining levels. The stress levels of the obtained strength points are given in Table 5.8.

![Figure 5.20: Stress path of test A11a](image-url)
Table 5.8: Peak points of the triaxial multi-stage strength test A11a

<table>
<thead>
<tr>
<th>Point</th>
<th>Confinement pressure [MPa]</th>
<th>Axial stress [MPa]</th>
<th>Axial strain(^1) [%]</th>
<th>Radial strain(^1) [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5.0</td>
<td>35.6</td>
<td>1.080</td>
<td>-0.382</td>
</tr>
<tr>
<td>B</td>
<td>7.5</td>
<td>42.5</td>
<td>1.165</td>
<td>—</td>
</tr>
<tr>
<td>C</td>
<td>10.0</td>
<td>50.0</td>
<td>1.282</td>
<td>—</td>
</tr>
<tr>
<td>D</td>
<td>12.5</td>
<td>56.6</td>
<td>1.368</td>
<td>—</td>
</tr>
</tbody>
</table>

\(^1\) referred to the start of the test

Figures 5.21 shows the stress-strain curve obtained for the four loading phases. It can be observed that it has been possible to determine with accuracy and univocally the peak values for each compression curve, as shown in Figure 5.22. It has not been possible to evaluate the radial strains for technical problems related to the local radial transducer.

Figure 5.23 shows the volumetric stress-strain curves. It is possible to observe that the volumetric stiffness of the material increases after the first shear phase and remains almost constant for the following three phases.

As shown in Figure 5.24 the failure of the specimen has developed along a well defined plane, inclined 58° on the horizontal plane.

Figure 5.21: Stress-strain curves for the shear phases of test A11a
5.8. Triaxial tests

Figure 5.22: Determination of the peaks of the stress-strain curves for the shear phase of test A11a

Figure 5.23: Volumetric stress-strain curves for the compression phases of test A11a
5.8.3 Test A17b

Test A17b is a multi-stage creep test performed in order to evaluate the time dependent behaviour of coal. The test can be subdivided into a consolidation and a shear phase.

Consolidation phase

The consolidation phase has been performed under isotropic conditions and with a constant stress rate of 0.14 MPa/min, up to a confining pressure of 10 MPa (stress path O→B of Figure 5.25).

Figure 5.26 shows the volumetric stress-strain curve obtained. The curve does not start from the origin because, for technical reasons, it has been necessary to increase the confining pressure up to 1 MPa without the control of the personal computer. If this curve is compared to the one of test A11a, it is possible to notice a similar trend but a little higher stiffness. This is probably due to the fact that block A17 has been taken from a greater depth than block A11.

The consolidation phase has been stopped at a confining pressure of 5 MPa (point A of Figure 5.25) and 10 MPa (point B of Figure 5.25) in order to perform
5.8. Triaxial tests

Figure 5.25: Stress path of test A17b

Figure 5.26: Volumetric stress-strain curves for the consolidation phase of test A17b
creep tests under volumetric stress conditions. The confining pressure has been kept constant and the local deformations of the specimen have been measured versus time. The creep phase at 5 MPa has been stopped only after few days due to problems with the triaxial apparatus. The creep curves obtained are depicted in Figure 5.27. The creep strains are small and tend to a constant value after a short period of time.

![Creep curves under volumetric state of stress of test A17b](image)

**Figure 5.27**: Creep curves under volumetric state of stress of test A17b

**Shear phase**

After consolidation a shear phase has been performed. The mean stress \( s \) has been kept constant while the deviatoric stress \( t \) has been increased with a constant rate of 0.16 MPa/min (Figure 5.25). A strength peak has been reached at the deviatoric stress \( t \) of 7.36 MPa (point E in Figure 5.25). The obtained stress-strain curve is illustrated in Figure 5.28.

The loading phase has been stopped for the deviatoric stress \( t \) of 5 MPa (point C in Figure 5.25) and 7 MPa (point D in Figure 5.25) in order to perform creep tests. The vertical load and the confining pressure have been held constant and the deformations of the sample have been measured versus time.

It is very interesting to observe the apparent preconsolidation phenomenon which has been discussed in Section 2.6.1 (Figure 5.29). When the specimen is reloaded after a creep phase it starts to deform with a higher stiffness than that of the compression phase prior to creep. It is possible to say that in this phase
5.8. Triaxial tests

Figure 5.28: Stress-strain curve for the shear phase of test A17b

the deformations are nearly elastic. The stiffness then decreases progressively until the point rejoins the original compression curve. At this point the stiffness returns equal to the stiffness before creep.

It is like to say that loading after creep cancels completely the “memory of creep” of the material; in fact the point rejoins the original compression curve and the stiffness returns to be equal to the value before creep. The sample seams to have a “short memory of creep”, and not a “long memory of creep”.

Figure 5.29: Apparent preconsolidation phenomenon for the shear phase of test A17b
The axial creep curves obtained are given in Figure 5.30. It is possible to notice that the strain rate, which is initially quite high, decreases rapidly with time (primary phase of creep, described in Section 2.3.2) and becomes almost constant after some time (secondary phase of creep, described in Section 2.3.2). It is important to observe that the magnitude of creep strains depends more than linearly from the applied deviatoric stress: an increase of only 40% of the applied deviatoric stress leads to an increase of 140% of the creep strains.

![Creep curve at t = 5 MPa (pt. C) and 7 MPa (pt. D) of test A17b](image)

**Figure 5.30:** Creep curve at t = 5 MPa (pt. C) and 7 MPa (pt. D) of test A17b

Figure 5.31 depicts the trend of the axial strain rate versus time in a logarithmic diagram. It is possible to observe a nearly linear relationship, as discussed in Section 2.3.3. The linear interpolation of the experimental data leads to values of $m$ close to 1: $m=0.81$ for the creep test at $t=5$ MPa (point C) and $m=1.07$ for the one at $t=7$ MPa (point D). Similar values were found by Tavenas et al. (1978).

Figure 5.32 shows the sample after test. It is possible to observe that the failure of the specimen developed along a well defined plane inclined approximately 47° to the horizontal plane.
5.8. Triaxial tests

Figure 5.31: Strain rate for the creep phase at $t = 5$ MPa (pt. C) and 7 MPa (pt. D) of test A17b

Figure 5.32: Photograph of the specimen A17b after the test
5.8.4 Test A17a

Test A17a has been performed with the intent to evaluate the time dependent behaviour of coal under higher confining pressure. It is a quite complex test as it is composed of a large number of phases (Figure 5.33). The goal is to obtain the maximum information from the same specimen.

It is possible to identify the following phases: (1) an isotropic consolidation phase including creep (stress path 0→A), (2) a shear phase with constant mean stress including creep (stress path A→E), (3) a stress relaxation test (stress path E→F), (4) a shear phase with constant confining pressure including creep (stress path F→K). Each phase is described in the following.

Figure 5.33: Stress path of test A17a

Consolidation phase

The consolidation phase (stress path 0→A in Figure 5.33) has been performed under isotropic conditions and with a constant stress rate of 0.1 MPa/min, up to the confining pressure of 20 MPa. The behaviour of the material during this phase is illustrated in Figure 5.34, where the volumetric stress-strain curve is reported. It is possible to observe a highly non-linear relationship between the confining pressure and the volumetric strain. This hardening behaviour is typical of granular soils. Comparing the obtained curve with the curves of the
consolidation phases of test A11a and test A17b a very similar trend is noted. Test A17b and A17a are characterized by a little higher stiffness than test A11a.

At the end of consolidation a creep phase has been performed: the confining pressure has been held constant and equal to 20 MPa and the local deformations of the sample have been measured versus time. No significant creep deformations have been observed during the 4 days of test.

**Shear phase at constant mean stress**

A shear phase with constant mean stress $s$ of 20 MPa has been performed up to a value of deviatoric stress $t$ of 13 MPa (stress path $A\rightarrow E$ in Figure 5.33). The applied loading rate is equal to 0.083 MPa/min.

The stress-strain curve obtained is illustrated in Figure 5.35. Comparing this curve with that of the shear phase of test A17b a higher stiffness is noted, which is probably due to the greater confining pressure.

The shear phase has been stopped for a deviatoric stress $t$ of 7 MPa (point C) and 10 MPa (point D) in order to perform creep tests. The deviatoric stress and the mean stress have been kept constant while the deformations of the sample have been measured. No significant creep deformations have been measured during the 16 days of the first creep test at $t=7$ MPa (point C). Only small creep deformations have been observed during the 50 days of the second creep
Figure 5.35: Stress-strain curve for the shear phase at constant mean stress (s.p. A→E) of test A17a

Figure 5.36: Creep curve at t=10 MPa (pt. D) of test A17a
5.8. Triaxial tests

test at $t=10$ MPA (point D), as shown in Figure 5.36. In this phase the deferred strain tends to a constant value after a short period of time. The fluctuation of the curve is probably due to a weekly variation of temperature inside the laboratory.

**Stress relaxation phase**

Immediately after the shear phase, a relaxation test has been carried out (stress path $E\rightarrow F$ of Figure 5.33). No movement of the loading frame has been allowed and the confining pressure has been kept constant. The change in the deviatoric load have been measured versus time. As shown in Figure 5.37, the total variation of the deviatoric stress $\sigma_a - \sigma_r$ is small. The stress rapidly decreases during the first days and then reaches a nearly constant value after 8 days. The fluctuation of the curve is probably due to a daily variation in temperature inside the laboratory. This test corroborates the hypothesis of the existence of a final relaxed state.

![Figure 5.37: Decrease of the deviatoric stress for the relaxation phase (s.p. E→F) of test A17a](image)

If the deviatoric stress is plotted versus the logarithm of time (Figure 5.38) it is possible to observe a nearly linear relationship that confirms the experimental evidences reported in literature (see Section 2.4.2).
Shear phase at constant confining pressure

After stress relaxation a new shear phase at constant confining pressure has been performed (stress path F→K in Figure 5.33). The confining pressure $\sigma_r$ has been kept constant at 7 MPa, while the deviatoric stress $\sigma_a - \sigma_r$ has been increased with a constant rate of axial displacement of 0.01 mm/min. The compression phase has been interrupted for the deviatoric stress $t$ of 14 MPa (point G), 15 MPa (point H), 16 MPa (point I), 17 MPa (point J) and 18 MPa (point K) in order to perform creep tests. The main intent of this phase is to evaluate the time dependent behaviour of coal approaching failure, with particular attention to the development of the tertiary phase of creep. The specimen has failed for creep rupture during the last creep phase.

Figure 5.39 shows the stress-strain curve obtained during this phase. The creep deformations are now significant if compared to the time independent deformations. The apparent preconsolidation effect in this case is not very clear, because of the strong non linearity of the compression curve near failure.

Figure 5.40 gives the creep curves obtained for the first four creep phases. The first two phases have been stopped only after 3 days for technical reasons. The fluctuation of the curve is essentially due to a variation of temperature inside the laboratory. The temperature measurement has highlighted a considerable daily variation of $\pm 1 \div 2 \, ^\circ$ with a more consistent increase during the week-end.
5.8. Triaxial tests

**Figure 5.39:** Stress-strain curve for the shear phase at constant confining pressure (stress path $F \rightarrow K$) of test A17a

**Figure 5.40:** Creep curve at $t = 14$ MPa (pt. G), 15 MPa (pt. H), 16 MPa (pt. I) and 17 MPa (pt. J) of test A17a
This unexpected variation is due to a problem of the air conditioning system of the laboratory that has been resolved before the start of the fifth creep phase (point K).

By observing the creep curves obtained it is possible to notice both the primary and the secondary phase of creep. The magnitude of creep increases following the increase of the applied deviatoric stress.

Figure 5.41 gives the axial strain rate versus time in the logarithmic diagram for the four creep phases. The tendency is almost linear and the interpolation of the experimental data leads to values very close to 1.

Figure 5.42 shows the creep curves obtained for the fifth creep phase at

Figure 5.41: Strain rate for the creep phase at $t = 14\, \text{MPa (pt. G)}$, 15 MPa (pt. H), 16 MPa (pt. I) and 17 MPa (pt. J) of test A17a
$t=18 \text{ MPa (point K). During this phase the tertiary phase of creep has developed}
and has led the specimen to failure in approximately 36 hours. It is possible to
clearly see the primary, secondary and tertiary phases of creep.

![Creep curve at $t=18 \text{ MPa (point K)}$ of test A17a](image)

*Figure 5.42: Creep curve at $t=18 \text{ MPa (point K)}$ of test A17a*

The axial strain rate of this phase is plotted versus time in Figure 5.43 in a logarithmic diagram. The minimum of the curve (reversal of acceleration) corresponds to the start of the tertiary phase of creep.

During the five creep phases also the radial displacement of the specimen could be measured, so that the relationship between the radial strain and the axial strain could be obtained. As easily seen in Figure 5.44 it is possible to identify a nearly linear trend.

As shown in Figure 5.45, where a photograph of the specimen after test is reported, the sample does not fail along a well defined plane.
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Figure 5.43: Strain rate for the creep phase at $t=18$ MPa (point K) of test A17a

Figure 5.44: Radial strains versus axial strains for the creep phases of test A17a
5.8.5 Test A17d

Test A17d has been performed under uniaxial conditions with the intent to evaluate the time dependent behaviour of coal at low confinement pressure. Particular attention has been posed on the development of the tertiary phase of creep. The stress path of the test is shown in Figure 5.46.

The axial stress has been increased with a constant displacement rate of 0.01 mm/min. The loading phase has been interrupted for the the axial stress $\sigma_a$ of 6 MPa (point A) and 8 MPa (point B) respectively, in order to perform creep tests: the axial load has been kept constant and the deformations of the specimen have been monitored versus time. During the second test a tertiary phase of creep developed and led the sample to failure in a quite short time.

The stress-strain curve obtained is shown in Figure 5.47. The apparent preconsolidation effect when the sample is reloaded can be observed after the first creep phase. If the stress-strain curve is compared with that for A11a, it is possible to observe a value of stiffness definitely smaller. This is probably due to the hardening of the material with the confining pressure (test A11a has been performed at 10 MPa of confining pressure).
Figure 5.46: Stress path of test A17d

Figure 5.48 gives the creep curve for the first creep phase (point A). It is possible to observe a behaviour which is similar to that observed in the previous creep tests: the strain rate decreases rapidly during the first part of the test (primary phase of creep) and then reaches a constant value (secondary phase of creep).

Figure 5.49 illustrates the creep curve obtained for the second creep phase (point B). It is possible to observe all the three phases of creep: primary, secondary and tertiary. In this case the tertiary phase of creep develops after a period of time (33 minutes) that is rather smaller than that of test A17a (36 hours). In fact, as reported in Section 2.3.5 the time to failure depends in a non linear manner from the applied deviatoric stress state.

The axial strain rate for the two creep phases is depicted in the logarithmic diagram of Figure 5.50. It is possible to observe a nearly linear behaviour of the first creep phase and the second creep phase before the development of the tertiary phase of creep.

As illustrated in Figure 5.51, which shows the specimen after the test, the rupture pervades the sample, and it is not possible to find a well defined shear plane of failure.
5.8. Triaxial tests

Figure 5.47: Stress-strain curve for the shear phase 0→B of test A17d

Figure 5.48: Creep curve at $\sigma_a=6$ MPa (point A) for test A17d
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Figure 5.49: Creep curve at $\sigma_a = 8$ MPa (point B) for test A17d

Figure 5.50: Strain rate for the creep phases of test A17d
5.9 Results

Two oedometer direct shear tests and four triaxial tests have been performed with the intent to evaluate:

- the strength characteristics of coal, with particular attention to peak and residual values;
- the time independent behaviour, with reference to volumetric and deviatoric stress conditions;
- the time dependent behaviour, by means of creep and relaxation tests; particular attention has been posed to the development of creep failure.

All the tests have been carried out using cylindrical specimens, that have been obtained with great difficulties and in a limited number. This has led to the necessity to obtain the maximum number of information from each sample. Therefore a great number of stages (strength, creep, relaxation) have been
performed for each test. This allows makes it difficult to interpret the results obtained.

The small number of tests performed is the most important limit of this work, especially if the heterogeneity of the sample is taken into account. In some occasions it is possible to define only a qualitative behaviour or trend for the material tested.

### 5.9.1 Strength behaviour

It has been possible to evaluate a significant number of strength points from the triaxial and direct shear tests on coal. The strength points obtained are given in Table 5.9 and depicted in Figure 5.52 in a \( t - s \) plane.

<table>
<thead>
<tr>
<th>Point</th>
<th>Test</th>
<th>Type</th>
<th>( s ) [MPa]</th>
<th>( t ) [MPa]</th>
<th>Interpolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A11a</td>
<td>triaxial peak strength</td>
<td>20.30</td>
<td>15.30</td>
<td>yes</td>
</tr>
<tr>
<td>B</td>
<td>A11a</td>
<td>triaxial peak strength</td>
<td>25.01</td>
<td>17.51</td>
<td>yes</td>
</tr>
<tr>
<td>C</td>
<td>A11a</td>
<td>triaxial peak strength</td>
<td>30.01</td>
<td>20.01</td>
<td>yes</td>
</tr>
<tr>
<td>D</td>
<td>A11a</td>
<td>triaxial peak strength</td>
<td>34.55</td>
<td>22.05</td>
<td>yes</td>
</tr>
<tr>
<td>E</td>
<td>A17b</td>
<td>triaxial peak strength</td>
<td>10.00</td>
<td>7.36</td>
<td>yes</td>
</tr>
<tr>
<td>F</td>
<td>A17a</td>
<td>triaxial creep rupture</td>
<td>25.00</td>
<td>18.00</td>
<td>no</td>
</tr>
<tr>
<td>G</td>
<td>A17d</td>
<td>triaxial creep rupture</td>
<td>4.00</td>
<td>4.00</td>
<td>no</td>
</tr>
<tr>
<td>H</td>
<td>A17c</td>
<td>shear peak strength</td>
<td>50.00</td>
<td>30.86</td>
<td>yes</td>
</tr>
<tr>
<td>I</td>
<td>A17e</td>
<td>shear peak strength</td>
<td>5.00</td>
<td>9.48</td>
<td>no</td>
</tr>
<tr>
<td>J</td>
<td>A17e</td>
<td>shear peak strength</td>
<td>15.00</td>
<td>16.08</td>
<td>no</td>
</tr>
<tr>
<td>K</td>
<td>A17e</td>
<td>shear residual strength</td>
<td>25.00</td>
<td>10.20</td>
<td>—</td>
</tr>
</tbody>
</table>

In order to evaluate the peak strength envelope of coal only the triaxial peak strength points (points A, B, C, D and E) and direct shear peak strength point of test A17c (point H) are considered. The triaxial creep rupture points (points G and F) are not taken into account because the creep rupture can develop far from the failure surface. The two peak points obtained from test A17e (points I and J) are not considered because they overestimate the strength of the material at low confining pressure. The experimental data are interpolated by using a linear Mohr-Coulomb function. The parameter obtained are given in Table 5.10.

The real strength envelope of the material is expected to be high non linear for small values of \( s \). Thus, the Hoek-Brown criterion could be more suitable to interpret the data. However, due to the small number of strength points, the lack of experimental data at low confinement stress, and the lack of tensile test
data, the Mohr-Coulomb criterion has been adopted.

![Graph showing peak and residual strength envelope of coal](image)

**Figure 5.52:** Peak and residual strength envelope of coal

**Table 5.10:** Mohr-Coulomb peak and residual strength parameter of coal

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friction angle $\phi$</td>
<td>34.29°</td>
</tr>
<tr>
<td>Cohesion $c$</td>
<td>3.52 MPa</td>
</tr>
<tr>
<td>Uniaxial strength $\sigma_{ci}$</td>
<td>13.31 MPa</td>
</tr>
<tr>
<td>Residual friction angle $\phi_r$</td>
<td>22.76°</td>
</tr>
</tbody>
</table>

The value of the residual friction angle $\phi_r$ has been evaluated by using the residual strength point of test A17e (point K) and by supposing that the residual strength envelope passes trough the origin.

Because of the small number of tests performed and the complexity of the time dependent creep tests, it has not been possible to evaluate the dependence of the strength envelope from the applied strain rate (see Section 2.5.1) or if a creep phenomenon can cause the hardening or the softening of the strength envelope (see Section 2.6.1).
5.9.2 Stress-strain behaviour

Volumetric behaviour

The high pressure oedometer tests do not permit to determine the preconsolidation pressure of coal using the classical Casagrande’s procedure.

Both the oedometer compression tests and the consolidation phases of the triaxial tests show a significant hardening of the material with the mean stress. If the loading modulus $E$ is plotted versus the mean stress $s$ (Figure 5.53) it is possible to observe that the loading modulus $E$ increases with the mean stress $s$ with a nearly linear relationship. This is a typical behaviour of a granular material (Lancellotta, 2004) and is probably due to the high porosity of coal.

![Figure 5.53: Loading modulus $E$ versus mean stress $s$ for the volumetric compression phases](image)

Shear behaviour

As shown in Figure 5.17, where the stress-strain curves of the shear phases of tests A17c and A17e are reported, the experimental direct shear behaviour of coal is nearly uniform and it is characterized by a mean stiffness of 400 MPa.
5.9. Results

Deviatoric behaviour

The mean stress $s$ seems to influence directly the loading modulus $E$: higher the means stress $s$, higher is the loading modulus $E$. This is a typical soil behaviour but cannot be quantified exactly because of the few and different triaxial tests performed. As an example Figure 5.54 compares different stress-strain curves at different mean stress.

![Figure 5.54: Influence of the mean stress $s$ on the stress-strain curve of triaxial tests](image)

Figure 5.55 gives the loading modulus $E$ versus the deviatoric stress normalized to failure, for the shear phases of the four triaxial tests performed. It is possible to observe that for tests A11a and A17d, which have been performed at constant confining pressure (increasing mean stress $s$), the loading modulus is almost constant or increases slightly. For tests A17b and A17a, that have been performed at constant mean stress $s$, the modulus is initially higher and then decreases progressively approaching failure.

It is very interesting to observe that all the tests, with the exception of test A17d, are characterized by a loading modulus $E$ which is much higher than the elastic modulus determined in ultrasonic tests (Section 5.5) which have a mean value of 1720 MPa. This is probably due to the hardening of the material with the confining pressure. This hypothesis is confirmed by test A17d, that has not been subjected to a consolidation phase, and that is characterized by a loading modulus very similar to the ultrasonic modulus.
5.9.3 Time dependent behaviour

The time dependent behaviour of coal has been investigated extensively by means of creep and relaxation tests performed under triaxial, oedometric and direct shear conditions.

Creep

Creep tests performed under oedometer test conditions (Figure 5.12) or during the consolidation phase of triaxial tests (Figure 5.27), show that creep deformations under volumetric state of stress are small compared to the creep deformations under a deviatoric state of stress, and tends to a constant value after a short period of time (only the primary phase of creep). This confirms the experimental evidences found in literature (Section 2.3.4), that state that the creep deformations depend essentially on the applied deviatoric state of stress.

Depending on the level of the applied stress it is possible to observe:

- no creep (points A and C of test A17a);
- only the primary creep phase (Figure 5.36);
- the primary and secondary creep phases (Figures 5.30, 5.40 and 5.48);
5.9. Results

- the primary, secondary and tertiary creep phases (Figures 5.42 and 5.49).

It is practically impossible to evaluate with accuracy the stress levels that define the transition between these different behaviours, because of the few tests performed and the complexity of each test.

The tertiary phase of creep can lead the sample to failure in a few minutes (Figure 5.42) or in some hours (Figure 5.49) depending on the distance from the failure surface. Also this relationship cannot be quantified exactly.

The experimental strain-time behaviour of the creep tests follows the experimental observations found in literature (see Section 2.3.3): with the exception of the tertiary phase of creep, if the creep strain rates are plotted versus time in a logarithmic diagram it is possible to observe a nearly linear trend (see Figures 5.31, 5.41 and 5.50) with a mean slope very close to 1 for all the curves. Likewise if the creep strains are plotted versus the logarithm of time (Figure 5.56) it is possible to observe a nearly linear relationship.

![Figure 5.56: Creep strain versus logarithm of time](image)

If the sample is reloaded after a creep phase it is possible to observe the apparent preconsolidation effect described in Section 2.6.1 (Figures 5.13, 5.29, 5.39 and 5.47).

Quite complex is to evaluate the stress dependency. In literature some authors found a non linear relationship between the logarithm of the strain rate.
and the applied deviatoric stress (see Section 2.3.4). A similar relationship is found to not hold true for the creep tests on coal.

For coal the magnitude of creep seems to depend on the distance from the failure surface, instead of the applied deviatoric stress. It the creep strains at fixed time are plotted versus the distance from the failure surface it is possible to find a nearly linear relationship, as shown in Figure 5.57. This observation is only qualitative because of the few tests performed and the complexity of each test. It should be very interesting to investigate in the future the validity of this by performing additional tests.

![Figure 5.57: Dependence of the axial strain from the distance to the failure surface for the triaxial creep tests on coal](image)

**Stress relaxation**

Only one relaxation test has been performed on coal. Therefore only few observations could be made on the relaxation behaviour. It is shown that the decrease of the deviatoric stress with time follows the experimental observations found in literature (see Section 2.4): if the deviatoric stress is plotted versus the logarithm of time it is possible to observe a linear behaviour as depicted in Figure 5.38. The stress decreases very rapidly in the first period of time and then reaches a final relaxed state.
5.10 Conclusions

The laboratory testing program described in the present chapter has been intended to point out the main characteristics of coal pertaining to the Carboniferous Formation met during the excavation of the Saint Martin La Porte access tunnel, which during excavation has shown a very severe squeezing behaviour. The fundamental aim of this study is to investigate the mechanical behaviour of coal which is supposed to widely influence the squeezing deformations of the rock mass. Particular attention has been posed on the time independent deformability, strength characteristic, and time dependent behaviour of the material.

Geotechnical classification highlights that coal can be considered as an intermediate material between soil and rock. Also the volumetric stress-strain curve observed under triaxial and oedometer test conditions shows a behaviour similar to that of soil: the material hardens with the mean stress.

The peak strength, which has been evaluated by means of triaxial and direct shear tests, increases with the mean stress and can be represented well by using the Mohr-Coulomb criterion. The non-linearity near the origin is not very clear because of data for small mean stress and under tension.

The time dependent behaviour of coal has been investigated extensively by means of creep and stress relaxation tests performed at different states of stress. It is noticed that the deferred behaviour is quite significant if compared to the instantaneous deformations.

A very important aspect of time dependence is the development of the tertiary phase of creep, that leads the sample to a delayed failure. It develops if the stress is sufficiently close to the failure envelope. This aspect of creep can influence significantly the stability of the excavation.

Most of the time dependent characteristics of coal follow with accuracy the experimental evidences reported in literature. Only the load dependency differs considerably. Because of the few test performed it is not possible to quantify exactly these observations.

In conclusion, it is possible to state that the mechanical time independent and time dependent behaviour has been identified with completeness at the laboratory scale. These observations form the background for the formulation of a new viscoplastic constitutive model to be discussed in the following chapter.

As it is obvious, the mechanical characteristics of coal cannot describe the behaviour of the Saint Martin La Porte tunnel, however they can be used as “guide lines” for the numerical analyses to be performed.
Chapter 6

The SHELVIP constitutive model

6.1 Introduction

The present chapter describes the novel elastoviscoplastic constitutive model SHELVIP (Stress Hardening ELastic VIscous Plastic), which is proposed with the intent to describe the squeezing phenomenon, that can occur during tunnel excavation in poor rock mass conditions (Barla, 2001, 2005).

The first part of the chapter explains the reasons for proposing a newly developed time dependent constitutive model, the background, the main hypotheses that have been made, the mathematical formulation, the fundamental characteristics, the implementation into a numerical code (FLAC), and its validation studies.

The second part of the chapter describes the application of the model to the laboratory tests that have been performed on coal, as described in Chapter 5.

6.2 Why a new constitutive model?

Tunnel construction in squeezing conditions is very demanding due to the difficulty in making reliable predictions at the design stage. Numerical analysis should improve the understanding of this phenomenon which is very complex and multiform, and can assist in the choice of the appropriate excavation and support systems to be adopted (e.g. mechanized tunnel is still an open question). Nowadays the main problem of numerical analysis is the choice of a suitable constitutive model that can correctly take into account the real behaviour of the rock mass being excavated (Barla, 2005).

Squeezing is often represented as an equivalent elastic-plastic medium with strength and deformability parameters which are down-graded, based on ob-
Chapter 6. The SHELVIP constitutive model

servation and monitoring during excavation. The so called “short term” and “long term” conditions are often invoked, characterized by different values of the parameters involved in the constitutive model being used. However, there is no doubt that under the most severe squeezing conditions an appropriate representation of the tunnel response is obtained only by using constitutive models which account explicitly for time dependent behaviour (Barla, 2001, 2005). This originates from the fact that time dependent deformations are observed whenever face advancement is stopped and these are likely to take place during excavation, when it is difficult to distinguish the “face effect” from the “time effect”.

The choice of a constitutive model is not easy at all: a very wide number of time dependent constitutive models have been proposed in literature to describe the deferred behaviour of geomaterials (see Section 2). They differ each other for the type of soil considered (clay, sand, rock), the main hypothesis, the aspects of creep taken into account, the in situ conditions, and the numerical formulation.

It is quite obvious that the perfect model does not exist. A suitable constitutive law must satisfy at least the following requirements: (1) it must describe all the principal aspects of soil behaviour that can play an important role during tunnel excavation; (2) it must be quite easy but not simplistic: a model which is too complex or too simple cannot be used with confidence both for research and for design practice. It is very problematic for a constitutive model to satisfy both requirements, because the simple introduction of the time variable multiplies by itself the number of constitutive parameters.

The fundamental aspects that a time dependent constitutive model is to take into account to correctly represent squeezing during tunnel excavation are:

- The behaviour near or at failure. The soil immediately around the excavation is frequently at failure or in near failure conditions. It is very important to correctly take into account the interaction between time dependent behaviour and failure. An example is the tertiary phase of creep that can develop near failure and lead the material to a delayed failure. It is also important to consider that a soil can yield instantly under the applied loads, even if the creep effects have not taken place yet.

- The long term behaviour. The time dependent behaviour of the soil is not always the same but depends on the loading history and on the previous deferred process that has taken place (i.e. it hardens). Moreover, the long term behaviour (few years) can be very different from the behaviour observed in laboratory tests or during the construction phases (few days or months).

- The stress-strain behaviour. Tunnel excavation is not only a deformability problem. It is a more complex problem of strong interaction between the
soil and the support system. Therefore, it is very important to correctly describe not only the time dependent deformations of the material, but also the relationship between stress and strain.

As already pointed out in Chapter 2, only few models in literature can correctly represent all these aspects. The main problem is that they are quite complex and a very large number of constitutive parameters are used, so they cannot fulfil the second requirement of simplicity. It is noteworthy to consider that in current engineering practice the most frequently used constitutive law is based on elastoplasticity, often associated with a linear non-hardening strength criterion (Mohr-Coulomb).

These considerations lead to the necessity to develop a new time dependent constitutive model that can be quite simple and, at same time, represent all the most important aspects of creep.

The proposed model would be a simple extension of the classical theory of elastoplasticity, by adding a time dependent component based on the overstress theory of Perzyna. It is important to highlight that the version of the SHELVIP model is currently under development and further improvement are in progress. It is expected that the tertiary phase of creep and creep damage will be introduced in the near future.

6.3 Background

Developing a time dependent constitutive model involves a wide number of aspects. It can be carried out from various points of view: mechanistic, phenomenological or purely theoretical. For everyone the fundamental goal is to propose a mathematical structure that allows one to reproduce the experimental behaviour of the soil observed at the laboratory scale and/or in situ.

The SHELVIP model has been formulated only from a phenomenological point of view: the interest is restricted only to factors that concern the macro-mechanical properties, such as stress, time, and strain. The micro-mechanical properties of the material, such void index, grain size, etc., are not considered.

Because the few tests performed on coal do not allow to obtain a complete and exhaustive grasp of the time dependent behaviour of coal, the formulation of the SHELVIP model has been based on various other elements. They are listed in the following in order of importance:

- Constitutive models proposed in literature. It is very important to consider the recent developments of time dependent numerical modelling in the formulation of the new model (see Section 3). If some hypothesis seems to work fine it can be inserted into the formulation. It is also very important
that the new model is in accordance with the existing framework of constitutive laws. The SHELVIP model couples the classical theory of elastoplasticity, that is commonly used in design of tunnels, with the theory of elastoviscoplasticity of Perzyna’s. The elements of novelty are few and in any case they are consistent with the main hypothesis of the two basic theories. The SHELVIP model is in some way inspired by the CVISC model (Section 3.3.1), by Lemaitre’s model (Section 3.4.1) and by Dafalias’ model (Section 3.5).

- Experimental observations found in literature. The study of the experimental observations reported in literature for various soils from different authors allows one to highlight a quite general pattern of time dependent behaviour. This pattern must be represented as much as possible by the new constitutive model.

- Theoretical requirements. The new model should satisfy a wide number of theoretical requirements. Some of these requirements are connected to the classical requirements of constitutive modelling (e.g. the determination principle, the material objectiveness principle, the local action principle). Other requirements are simply due to logical or consistency principles (e.g. the deformations of the material cannot tend to infinite for infinite time, etc.).

- Numerical requirements. It is very important to consider also the requirements and the demands of numerical analysis. The model should be as simple as possible, the number of constitutive parameters should be limited, each parameter should have a physical meaning and should be determined in a simple manner from laboratory tests. The implementation into a numerical code should be simple, numerically stable and efficient.

- Experimental evidences from tests on coal. This element is placed at the end in order of importance, because it is very difficult to define a general and comprehensive pattern of time dependent behaviour of coal.

### 6.4 Formulation of the model

This section describes the main hypotheses and the mathematical formulation of the SHELVIP model, which has been formulated for simplicity reasons with reference to a homogeneous, perfectly isotropic, continuum medium.

The interstitial water pressure effect is taken into account by means of the classical Terzaghi’s effective stress principle. In the following one makes always reference to the effective state of stress, even if the notation is not explicit.
As mentioned above, the SHELVIP model combines together the classical theory of elasticity, the classical theory of plasticity, and the viscoplastic overstress theory of Perzyna (Perzyna, 1966). In this way, it is possible to take into account the instantaneous irreversible deformations, which are very important for tunnel excavation and, at the same time, to use a plastic law which is widely employed in design practice.

6.4.1 Strain partition

The main hypothesis of the SHELVIP model is that the total strain tensor can be split into an elastic, a plastic (time independent) and a viscoplastic (time dependent) component, to give:

\[ \varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p + \varepsilon_{ij}^{vp} \]  

(6.1)

where the apex \( e \) refers to the elastic component, the apex \( p \) to plastic component and the apex \( vp \) to the viscoplastic component. This relation holds true also for the strain rate tensor. If Equation 6.1 is derived over time, it is possible to obtain:

\[ \dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p + \dot{\varepsilon}_{ij}^{vp} \]  

(6.2)

6.4.2 Limit yield surfaces and stress fields

In the principal triaxial stress space two limit surfaces are defined by using the Drucker-Prager’s criterion: a plastic yield surface and a viscoplastic yield surface (Figure 6.1). The plastic surface defines the stress states of developments of plastic strains, according to the classical theory of elastoplasticity. The viscoplastic surface, which is internal to the plastic surface, defines the lower limit for development of time dependent deformations, according to the viscoplastic theory of Perzyna.

This two yield surfaces define the existence of three stress fields in the stress space:

- **an elastic field** inside the viscoplastic surface. The deformations are only elastic:

\[ \varepsilon_{ij} = \varepsilon_{ij}^e \]  

(6.3)

and can be calculated by using the classical theory of elasticity;

- **an elastic-viscoplastic field** between the viscoplastic surface and the plastic surface. The deformations are elastic and viscoplastic:

\[ \varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^{vp} \]  

(6.4)
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Figure 6.1: Limit surfaces and stress fields

Figure 6.2: Limit surfaces and stress fields in the $q - p$ plane
6.4. Formulation of the model

While the elastic strains $\varepsilon_{ij}^e$ can still be calculated by using the classical theory of elasticity, the viscoplastic strains $\varepsilon_{ij}^{vp}$ are determined by using the general flow rule of viscoplasticity;

- an **elastic-plastic-viscoplastic field** on the plastic surface. The deformations are elastic, plastic and viscoplastic:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p + \varepsilon_{ij}^{vp} \quad (6.5)$$

While the elastic strains $\varepsilon_{ij}^e$ and the viscoplastic strains $\varepsilon_{ij}^{vp}$ are still calculated by using elasticity and viscoplasticity, the plastic strains $\varepsilon_{ij}^p$ are evaluated by using the classical flow rule of plasticity.

The existence of a purely elastic field is very important in order to correctly model the loading conditions at low stress level and the unloading conditions.

In the following the mathematical expressions of the plastic and viscoplastic yield surfaces are illustrated with reference to the $q - p$ stress plane:

$$p = \frac{1}{3} \cdot \sigma_{mm} \quad (6.6)$$

$$q = \sqrt{\frac{3}{2} s_{ij} \cdot s_{ij}} \quad (6.7)$$

where $s_{ij}$ is the stress deviator tensor:

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \cdot \delta_{ij} \cdot \sigma_{mm} \quad (6.8)$$

**Plastic yield surface**

The plastic yield surface $f_p = 0$ is defined into the $q - p$ stress plane by using a linear Drucker-Prager criterion (Figure 6.2):

$$f_p = q - \alpha_p \cdot p - k_p = 0 \quad \text{for} \quad p \geq \sigma_t \quad (6.9)$$

where $\alpha_p$ and $k_p$ are the slope and the intercept with the $q$-axis of the linear criterion respectively, and can be derived from the strength parameters of the Mohr-Coulomb criterion using the following relationships for the circumscribing and inscribing cases:
circumscribing: \[
\begin{align*}
\alpha_p &= \frac{6}{3 - \sin(\phi)} \cdot \sin(\phi) \\
k_p &= \frac{6}{3 - \sin(\phi)} \cdot c \cdot \cos(\phi)
\end{align*}
\] (6.10)

inscribing: \[
\begin{align*}
\alpha_p &= \frac{6}{3 + \sin(\phi)} \cdot \sin(\phi) \\
k_p &= \frac{6}{3 + \sin(\phi)} \cdot c \cdot \cos(\phi)
\end{align*}
\] (6.11)

where \(c\) is the cohesion and \(\phi\) the friction angle.

The plastic surface is characterized by a volumetric tension cut-off \(\sigma_t\).

The plastic yield surface is always fixed and cannot harden neither during the plastic strain development nor during the viscoplastic strain development. The assumption of a fixed plastic surface is a useful simplification. Actually, the experimental observations reported in Chapter 2 show that the yield surface can expand or contract as a function of the applied strain rate or the deferred process that takes place. This phenomenon should be taken into account in further improvements of the model.

**Viscoplastic yield surface**

The viscoplastic yield surface \(f_{vp} = 0\) is also defined in the \(q - p\) stress plane by a linear Drucker-Prager criterion (Figure 6.2). It is internal to the plastic yield surface. It intersects the \(p\)-axis at the same point (Point O of Figure 6.2) as the plastic yield surface and is characterized by the same volumetric tension cut-off \(\sigma_t\). This allows to have two homothetic surfaces and permits to reduce the overall number of constitutive parameters.

Therefore the viscoplastic yield surface can be expressed by the relationship:

\[
f_{vp} = q - \alpha_{vp} \cdot \left( p + \frac{k_p}{\alpha_p} \right) = 0 \quad \text{for} \quad p \geq \sigma_t
\] (6.12)

by using only one parameter, \(\alpha_{vp}\), that defines the slope of the straight line in the \(q - p\) plane.

As discussed above, the viscoplastic yield surface defines the transitional stress level between the purely elastic and the elastic-viscoplastic behaviour of the material. Because this stress level can change during the loading history of the material, it is assumed that the viscoplastic yield surface can modify its position (it hardens) by means of the hardening law described in Section 6.4.6. Therefore the parameter \(\alpha_{vp}\), that defines the position of the viscoplastic yield surface, can be considered as an intrinsic state parameter of the viscoplastic hardening of the material (i.e. it is not a constitutive parameter of the model).
6.4.3 Elastic law

The elastic strain rate component $\dot{\varepsilon}_{ij}$ of Equation (6.2), which is always present, can be determined from the stress rate tensor $\dot{\sigma}_{kl}$ by using a linear elastic law:

$$\dot{\varepsilon}_{ij} = C_{ijkl} \cdot \dot{\sigma}_{kl}$$  \hspace{1cm} (6.13)

where $C_{ijkl}$ is the compliance matrix, which is constant and is defined by using the Young’s modulus $E$ and the Poisson’s ratio $\nu$ as follows:

$$C_{ijkl} = \frac{1}{E} \cdot [(1 + \nu) \cdot \delta_{ik} \cdot \delta_{jl} - \nu \cdot \delta_{ij} \cdot \delta_{kl}]$$  \hspace{1cm} (6.14)

6.4.4 Plastic flow rule

If the current state of stress is represented by a point on the plastic yield surface (i.e. $f_p(\sigma_{ij}) = 0$), plastic deformations develop and the plastic strain rate tensor $\dot{\varepsilon}^p_{ij}$ can be determined by using the classical flow rule of the theory of elastoplasticity:

$$\dot{\varepsilon}^p_{ij} = \lambda \cdot \frac{\partial g_p}{\partial \sigma_{ij}}$$  \hspace{1cm} (6.15)

where $\lambda$ is the so-called plastic multiplier, that can be determined using the consistency condition (i.e during plastic flow $f_p = 0$ and $\dot{f}_p = 0$), and $g_p$ is the plastic potential function that defines the direction of the plastic strain rate tensor.

The plastic flow is non-associated and the plastic potential function $g_p$ is assumed to be a linear function of the deviatoric stress $q$ and of the volumetric stress $p$ as follows:

$$g_p = q - \omega_p \cdot p$$  \hspace{1cm} (6.16)

where $\omega_p$ is the plastic dilatancy, which defines the ratio of volumetric to deviatoric plastic strain increments. Only if volumetric tensile yielding occurs the plastic flow is associated and the plastic potential function is equal to the plastic yield function $g_p = f_p$.

6.4.5 Viscoplastic flow rule

If the stress state exceeds the viscoplastic yield surface (i.e. $f_{vp}(\sigma_{ij}) > 0$) viscoplastic deformations develop. The viscoplastic strain rate tensor $\dot{\varepsilon}^{vp}_{ij}$ can be evaluated by using the flow rule of the overstress theory of Perzyna (Equation (3.55)): 

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\[ \dot{\varepsilon}_{ij}^{vp} = \gamma \cdot \Phi (F) \cdot \frac{\partial g_{vp}}{\partial \sigma_{ij}} \] (6.17)

where \( \gamma \) is a fluidity parameter, which controls the amplitude of the viscoplastic strain rate, \( \Phi (F) \) is the so-called viscous nucleus, \( F \) is the overstress function, \( g_{vp} \) is the viscoplastic potential function.

It is noted that the overstress function \( F \) represents the state of overstress inside the material with respect to a “static” condition, which is defined by the viscoplastic yield surface. It quantifies the distance between the stress point and the viscoplastic yield surface. In the SHELVIP model the overstress function is assumed to be equal to the viscoplastic yield function:

\[ F = f_{vp} \] (6.18)

and defines the deviatoric state of stress which exceeds the viscoplastic yield surface, as depicted in Figure 6.2 on the \( q - p \) plane.

The viscoplastic nucleus \( \Phi (F) \) controls the magnitude of the viscoplastic strain rate. It is assumed to be a power function of the overstress function \( F \):

\[ \Phi (F) = \langle F \rangle^n \] (6.19)

where \( n \) is a constitutive parameter \((n > 0)\). The Macauly brackets \( \langle \rangle \) \((\langle x \rangle = 0 \) if \( x < 0 \) and \( \langle x \rangle = 0 \) if \( x \geq 0 \)) allow for viscoplastic deformations only if the stress point is external to the viscoplastic yield surface.

The viscoplastic potential function \( g_{vp} \) defines the direction of the viscoplastic strain rate tensor. It is assumed to be a linear function of the deviatoric stress \( q \) and of the volumetric stress \( p \):

\[ g_{vp} = q - \omega_{vp} \cdot p \] (6.20)

where \( \omega_{vp} \) is the plastic dilatancy, which quantifies the ratio between the volumetric and the deviatoric viscoplastic strain increment.

With the assumptions above and given that:

\[ \frac{\partial q}{\partial \sigma_{ij}} = \frac{3}{2} \cdot \frac{s_{ij}}{q} \quad \text{and} \quad \frac{\partial p}{\partial \sigma_{ij}} = \frac{1}{3} \cdot \delta_{ij} \] (6.21)

Equation (6.17), which controls the evolution of viscoplastic strain rate, can be rewritten as (see Appendix B):

\[ \dot{\varepsilon}_{ij}^{vp} = \gamma \cdot \langle f_{vp} \rangle^n \cdot \left( \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right) \] (6.22)
6.4.6 Viscoplastic hardening

The additional element to be defined, in order to complete the formulation of the model, is the hardening rule for the viscoplastic yield surface. It is of course the most important component and the most significant element of novelty of the SHELVIP constitutive model. In fact, the previous assumptions made by different authors are simply the classical postulates of plasticity and viscoplasticity.

The hardening rule in nearly all the models presented in Chapter 3 is controlled by the viscoplastic strains, by means of a scalar quantity, like the viscoplastic deviatoric strain $\varepsilon_{vp}^{q}$ of Equation (3.56) or the viscoplastic work $W_{vp}$ of Equation (3.58). In our opinion the main problem of this approach is that the viscoplastic hardening level of the material can be defined only by means of these quantities, which are practically impossible to be determined both at the laboratory and in situ scale (only an indirect evaluation is possible). Moreover these quantities are not absolute but are defined with reference to an initial time value $t_{0}$. Therefore, they cannot be rigorously considered as “state variables” of the material.

In order to overcome these limitations a stress based hardening rule is assumed as a relationship between the time derivative of the parameter $\alpha_{vp}$, that defines the hardening level of the viscoplastic yield surface $f_{vp} = 0$, and the state of stress $f_{vp}(\sigma_{ij})$ that exceeds the viscoplastic yield surface. The proposed hardening rule can be expressed as:

$$\dot{\alpha}_{vp} = \frac{l}{m \cdot n} \cdot \frac{f_{vp}}{p + \frac{k_{p}}{\alpha_{p}}} \cdot \left(\frac{f_{vp}}{q}\right)^{m-n}$$  \hspace{1cm} (6.23)

where $l$ and $m$ are constitutive parameters ($l > 0$ and $m > 0$).

This is like to state that the existence of a deviatoric state of overstress inside the material for a finite period of time leads to a fixed variation of the viscoplastic yield function, which depends on the overall state of stress defined by $q$ and $p$.

Equation (6.23) is apparently complex. However, with the overstress theory of Perzyna holding true, the hardening law of the viscoplastic yield function drives at the same time the evolution of the viscoplastic strains versus time and the evolution of the stress levels for onset of viscoplastic deformations. The physical meaning of the proposed equation becomes self-evident if consideration is given to creep behaviour as discussed below.

The introduction of a stress based hardening law is associated with some advantages. The most important one is the possible evaluation of the viscoplastic hardening level of the material from the stress level which defines the threshold for development of viscoplastic deformations, which can be done by appropriate
tests. A second one is a clear definition of each time dependent feature by means of a single constitutive parameter. However, a disadvantage is found in the need to introduce an additional parameter ($l$) in order to satisfy the dimensional equality between the two terms of Equation (6.23).

Considering the viscoplastic law given by Equation (6.22) and the hardening law of Equation (6.23), it is possible to state that the parameter $\alpha_{vp}$ summarizes by itself all the previous loading history of the material.

### 6.4.7 Constitutive parameters

With the assumptions described above, the overall number of constitutive parameters of the SHELVIP model is 11. It is possible to note that it is a rather limited number, especially if the introduction of the time variable and the complexity of the model are considered. The constitutive parameters can be distinguished into elastic (2), plastic (4) and viscoplastic (5).

#### Elastic parameters
- $E$, elastic modulus;
- $\nu$, Poisson’s ratio.

They can be easily determined at laboratory scale by using classical triaxial tests. An unloading test phase performed at high constant strain rate is recommended in order not to take into account plastic or viscoplastic effects.

#### Plastic parameters
- $\alpha_p$, slope of the Drucker-Prager’s plastic yield criterion into the $q - p$ plane. It can be easily determined from the Mohr-Coulomb parameters, cohesion $c$ and friction angle $\phi$, by using Equations (6.10) and (6.11);
- $k_p$, intercept of the Drucker-Prager’s plastic yield criterion with the $q$-axis. It can be easily determined from the Mohr-Coulomb parameters, cohesion $c$ and friction angle $\phi$, by using Equations (6.10) and (6.11);
- $\sigma_t$, volumetric tension cut-off. It defines the tension cut-off of the Drucker-Prager’s plastic yield criterion into the $q - p$ plane;
- $\omega_p$, plastic dilatancy. It defines the ratio between the volumetric plastic strain increment $\Delta \varepsilon_p^v$ and the deviatoric plastic strain increment $\Delta \varepsilon_q^p$. 

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6.5 Analytical solution for creep

Viscoplastic parameters

- $\gamma$, fluidity parameter. It defines the amplitude of the viscoplastic strains;
- $m$, shape factor. It defines the shape of the creep curves;
- $n$, load factor. It defines the dependency of the viscoplastic strain rates from the deviatoric stress;
- $l$, time stretching factor. It allows to scale the time dependent curves with time;
- $\omega_{vp}$, viscoplastic dilatancy. It defines the ratio between the volumetric viscoplastic strain increment $\Delta \varepsilon_{vp}^v$ and the deviatoric viscoplastic strain increment $\Delta \varepsilon_{vp}^q$.

The effects of viscoplastic parameters on the time dependent behaviour of the model will be discussed further with reference to creep conditions. By observing the response of the model to a constant state of stress it is possible to notice that each aspect of creep is controlled by a single constitutive parameter. This makes it easy to determine the constitutive parameters and perform numerical back analysis.

6.4.8 State parameters

In addition to the classical state parameters of continuum mechanics (stress, strain, etc.) the SHELVIP model provides an additional scalar state parameter $\alpha_{vp}$, called viscoplastic hardening level. It defines the stress level for onset of viscoplastic deformations and takes into account the complete stress-strain-time loading history of the material.

6.5 Analytical solution for creep

If creep conditions are considered, an analytical closed-form solution can be derived from the differential equations of the SHELVIP model. This is important as the constitutive parameters can be determined by using this solution and the experimental results from laboratory creep tests. Also, in this way a better understanding of this model of behaviour can be achieved.

Consider a triaxial creep test characterized by a constant axial stress $\sigma_a$ and a constant radial stress $\sigma_r$, which define a point in the $q - p$ stress plane ($q = \sigma_a - \sigma_r$, $p = (\sigma_a + 2 \cdot \sigma_r)/3$) located between the viscoplastic yield surface and the plastic yield surface. With this holding true, the behaviour of the material is elasto-viscoplastic. The complete derivation is given in Appendix B.
If the time derivative of the viscoplastic function given by Equation (6.12) is introduced into Equation (6.23), the hardening law of the viscoplastic yield surface can be rewritten as:

\[
\dot{f}_{vp} = -\frac{l}{m \cdot n} \cdot f_{vp} \cdot \left( \frac{f_{vp}}{q} \right)^{m-n}
\]

(6.24)

For small intervals of time, the decrease of the state of overstress inside the material, which is caused by the hardening of the viscoplastic yield surface, is directly proportional to the state of overstress itself multiplied by a factor, which takes into account the ratio between the state of overstress and the overall deviatoric state of stress.

Equation (6.24) can be easily integrated, leading to:

\[
f_{vp} = q \cdot \left[ l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^{m-n} \right]^{-\frac{1}{m-n}}
\]

(6.25)

where \( f_{vp,0} \) is the initial value of the viscoplastic function, given by Equation (6.12), with the initial viscoplastic hardening level equal to \( \alpha_{vp,0} \):

\[
f_{vp,0} = q - \alpha_{vp,0} \cdot \left( p + \frac{k_p}{\alpha_p} \right)
\]

(6.26)

If Equation (6.25) is introduced into Equation (6.22), it is possible to derive the axial viscoplastic strain rate \( \dot{\varepsilon}_{vp} \) in analytical form as follows:

\[
\dot{\varepsilon}_{vp} = \gamma \cdot q \cdot n \cdot \left[ l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^{m-n} \right]^{-\frac{1}{n}} \cdot \left( 1 - \frac{\omega_{vp}}{3} \right)
\]

(6.27)

By integrating Equation (6.27) over the time variable it is possible to obtain two different expressions for the radial viscoplastic strain \( \varepsilon_{vp} \): one for \( m \neq 1 \), and another one for \( m = 1 \).

The expression for \( m \neq 1 \) is:

\[
\varepsilon_{vp} = \frac{\gamma}{l} \cdot \frac{m}{m-1} \cdot q^n \left\{ l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^{m-n} \right\}^{\frac{m-1}{m}} - \left( \frac{q}{f_{vp,0}} \right)^{n(m-1)} \left( 1 - \frac{\omega_{vp}}{3} \right)
\]

(6.28)

The expression for \( m = 1 \) is:

\[
\varepsilon_{vp} = \frac{\gamma}{l} \cdot q^n \cdot \ln \left[ 1 + l \cdot t \cdot \left( \frac{f_{vp,0}}{q} \right)^n \right] \cdot \left( 1 - \frac{\omega_{vp}}{3} \right)
\]

(6.29)

Equations (6.28) and (6.29) lead to underline some of the most relevant behavioural features of the SHELVIP model as discussed in the following.
6.5. Analytical solution for creep

6.5.1 Strain-time behaviour

Based on Equation (6.29), the SHELVIP model incorporates for the semi-logarithmic law of creep:

\[
\varepsilon_{a}^{vp} = C_{\alpha} \cdot \ln \left( 1 + \frac{t}{t_{0}} \right)
\]

(6.30)

with:

\[
t_{0} = \frac{1}{l} \cdot \left( \frac{q}{f_{vp,0}} \right)^{n} \quad \text{and} \quad C_{\alpha} = \frac{\gamma}{l} \cdot q^{n} \cdot \left( 1 - \frac{\omega_{vp}}{3} \right)
\]

(6.31)

where \(t_{0}\) is the reference time and \(C_{\alpha}\) is the secondary compression coefficient for triaxial creep tests.

The model can correctly take into account the dependency of the reference time \(t_{0}\) from the loading history of the material. If the initial hardening level \(\alpha_{vp,0}\) is equal to zero (i.e. the material did not experience any creep process in the past) the reference time is equal to the inverse of \(l\); otherwise creep strains develop at a lower rate. This allows one to correctly represent a creep process which has started in the past. Furthermore, the model can describe the dependency of the secondary compression coefficient \(C_{\alpha}\) from the applied deviatoric stress \(q\), by means of the constitutive parameter \(n\).

The existence of a different solution for creep strain for \(m = 1\) suggests that this particular value should be considered as a critical value that separates two different models of behaviour. If the limit of the axial viscoplastic strain \(\varepsilon_{a}^{vp}\) of Equations (6.28) and (6.29) for time tending to infinity is evaluated, it is possible to obtain:

\[
\left\{ \begin{array}{ll}
\lim_{t \to \infty} \varepsilon_{a}^{vp} &= \frac{\gamma}{l} \cdot \frac{m}{m-1} \cdot q^{n} \cdot \left( \frac{q}{f_{vp,0}} \right) \cdot \left( 1 - \frac{\omega_{vp}}{3} \right) & \text{if} \quad m < 1 \\
\lim_{t \to \infty} \varepsilon_{a}^{vp} &= \infty & \text{if} \quad m \geq 1
\end{array} \right.
\]

(6.32)

If \(m < 1\) the axial viscoplastic strain tends to a constant value for time tending to infinity. Therefore the model is able to describe only the primary phase of creep (Figure 6.3). If \(m \geq 1\) the axial viscoplastic strain tends to infinity and the model can describe also the secondary phase of creep (Figure 6.3). It should be noted that the SHELVIP model cannot describe the tertiary phase of creep.

Rigorously, only the condition \(m < 1\) can be considered as physically admissible, because strain cannot exceed the limit of 100% in compression. However, the case \(m > 1\) can better describe the short-term and the medium-term behaviour. On the contrary, if the long-term behaviour is considered, the condition \(m < 1\) will be strictly required.
6.5.2 Strain rate-time behaviour

Some important considerations can be done by investigating the creep strain rate-time behaviour of the SHELVIP model, which is governed by Equation (6.27).

If the logarithm of the viscoplastic strain rate $\dot{\varepsilon}_{vp}$ is plotted versus the logarithm of time $t$ it is possible to obtain the curve shown in Figure 6.4. This behaviour is characterized by two different asymptotes, one horizontal on the left (called “short-term asymptote”) and one inclined on the right (called “long-term asymptote”), which are joined one to the other one by a curved line.

The limit of the viscoplastic strain rate for time tending to zero is equal to a constant value, and gives the horizontal asymptote on the left:

$$\lim_{t \to 0} \dot{\varepsilon}_{vp} = \dot{\varepsilon}_{vp}^{\infty} = \gamma \cdot f_{vp,0} \cdot \left(1 - \frac{\omega_{vp}}{3}\right)$$

(6.33)

which is shown to considerably improve the numerical stability and the accuracy of the constitutive model. It prevents the strain rate to tend to infinity for time tending to zero, which is a common problem of full-logarithmic models. Although this asymptote is not clearly evidenced from results of testing, it does not influence significantly the shape of the creep curves, because it involves only the very first instants of time.

The limit of the viscoplastic strain rate for time tending to infinity gives the inclined asymptote on the right:
6.5. Analytical solution for creep

Figure 6.4: Strain rate-behaviour on a logarithmic diagram

\[
\lim_{t \to \infty} \dot{\varepsilon}^{vp}_{a} = \dot{\varepsilon}^{vp}_{a}\big|_{\infty} = \gamma \cdot l^{-\frac{1}{m}} \cdot t^{-\frac{1}{m}} \cdot q^{n} \cdot \left(1 - \frac{\omega_{vp}}{3}\right) \quad (6.34)
\]

which can be rewritten into logarithmic form as follows:

\[
\log(\dot{\varepsilon}^{vp}_{a}\big|_{\infty}) = -\frac{1}{m} \cdot \log(t) + n \cdot \log(q) + \log \left[\gamma \cdot l^{-\frac{1}{m}} \cdot \left(1 - \frac{\omega_{vp}}{3}\right)\right] \quad (6.35)
\]

Thus a linear relationship between the logarithm of viscoplastic strain rate and the logarithm of time, with a slope equal to \(1/m\), is obtained as shown in Figure 6.4. The SHELVIP model is found to reproduce satisfactorily the results of tests reported by different authors (Singh and Mitchell, 1968; Bishop and Lovenbury, 1969; Tavenas et al., 1978), except for very small values of time. In fact, these tests exhibit a linear relationship between creep strain rate and time into a logarithmic diagram, when tertiary creep is not present.

It is important to observe that the parameter \(m\) of the SHELVIP model is equal to the inverse of the parameter \(m\) defined by Singh and Mitchell (1968).

It is noted that an equation similar to Equation (6.35) can also be obtained for the radial viscoplastic strain rate \(\dot{\varepsilon}^{vp}_{r}\), so that one may state that the parameter \(m\) characterises at the same time the evolution of both the volumetric and the deviatoric creep strains as assumed by Tavenas et al. (1978). This applies to
the majority of geomaterials, although in cases it may not hold true as reported by Feda (1992) and Tian et al. (1994).

In the SHELVIP model the parameter \( m \) is independent from the applied deviatoric stress, as shown by Tavenas et al. (1978). However, some authors report that \( m \) is not always independent from the deviatoric stress: in some cases \( m \) is found to decrease (Bishop and Lovenbury, 1969; Tian et al., 1994; Feda, 1992); in other cases the opposite is true (Zhu et al., 1999). Given this uncertainty, the assumption that \( m \) is independent from the deviatoric stress may be acceptable.

### 6.5.3 Stress dependency

If the initial viscoplastic hardening \( \alpha_{vp,0} \) is assumed to be equal to zero (i.e. the material is not yet affected by time dependent phenomena), the initial viscoplastic overstress function \( f_{vp,0} \) is equal to the deviatoric stress \( q \) (Equation (6.26)) and the equation of the axial viscoplastic strain rates (Equation (6.27)) can be rewritten to:

\[
\dot{\varepsilon}_{vp}^a = \gamma \cdot q^n \cdot (l \cdot t + 1)^{-\frac{1}{n}} \cdot \left(1 - \frac{\omega_{vp}}{3}\right)
\]  

(6.36)

It is noted that the viscoplastic strain rate and, as a consequence, the viscoplastic strain given in such a condition by the SHELVIP model depend only on the deviatoric stress \( q \). Then the volumetric stress \( p \) does not influence the development of viscoplastic deformations. This particular aspect of the time dependent behaviour is commonly accepted and confirmed by various tests.

According to Equation (6.36), a linear relationship exists between the logarithm of the viscoplastic strain rate \( \dot{\varepsilon}_{vp}^a \) and the logarithm of deviatoric stress \( q \) with a slope equal to the constitutive parameter \( n \). If the logarithm of the viscoplastic strain rates \( \dot{\varepsilon}_{vp}^a \) is plotted versus the applied deviatoric stress \( q \) the curve depicted in Figure 6.5 can be obtained. It is noteworthy that this curve is very similar to the curve of Figure 2.12 and Figure 2.13.a proposed by Mitchell (1993) in order to describe the stress dependence.

The SHELVIP model can describe very well the fact that at low stress the viscoplastic strain rates are small and the fact that in the midrange of stress a nearly linear relationship can be found between the logarithm of strain rate and the deviatoric stress. However the SHELVIP model cannot represent the rapid increase of the viscoplastic strain rate when stress approaches failure (tertiary phase of creep).
6.5. Analytical solution for creep

6.5.4 Initial viscoplastic hardening

It is of interest to analyse the effect of the initial viscoplastic hardening level \( \alpha_{vp,0} \) on both the viscoplastic strain rate and the viscoplastic strain during a creep test. To this purpose Equation (6.27) can be rewritten as:

\[
\dot{\varepsilon}_{vp} = \gamma \cdot \left[ q \cdot \left( \frac{f_{vp,0}}{q} \right) \right]^n \cdot \left[ \left( \frac{f_{vp,0}}{q} \right)^{m-n} \cdot l \cdot t + 1 \right]^{-\frac{1}{m}} \cdot \left( 1 - \frac{\omega_{vp}}{3} \right)
\]  

(6.37)

The increase of the initial viscoplastic hardening level \( \alpha_{vp,0} \) causes the initial overstress function \( f_{vp,0} \) to decrease according to Equation (6.26). This leads to two important consequences, as observed by comparing Equations (6.36) and (6.37). The first consequence is that the viscoplastic strain rate depends on the deviatoric stress \( q \), which is reduced by a factor \( \left( f_{vp,0}/q \right) \) smaller than 1, if compared to the case of initial viscoplastic hardening equal to zero. The second consequence is that the time \( t \) is multiplied now by a factor \( \left( f_{vp,0}/q \right)^{m-n} \), which is smaller than 1. This implies that the change in viscoplastic strain takes place at a lower rate with respect to the case of initial viscoplastic hardening equal to zero.

If the logarithm of the viscoplastic strain rate is plotted versus the logarithm of time, the increase of the initial overstress level \( \alpha_{vp,0} \) is observed to cause the short-term asymptote (Equation (6.33)) to move downwards without influencing
Figure 6.6: Effect of initial viscoplastic hardening to the strain rate-time curve of creep test

Figure 6.7: Effect of initial viscoplastic hardening to the strain-time curve of creep test
the long-term asymptote (Equation (6.35)), as depicted in Figure 6.6. This means that the increase of the initial viscoplastic hardening level $\alpha_{vp,0}$ influences only the short-term behaviour, and not the long-term behaviour, which is controlled by the deviatoric stress $q$.

The effect of an increase in the initial viscoplastic hardening level $\alpha_{vp,0}$ on the viscoplastic strain-time response is to bring the creep curves down, as illustrated in Figure 6.7. Thus, the magnitude of creep is reduced, but the shape of the curves remain nearly the same.

### 6.5.5 Geometrical effects of parameters

It is useful to observe the geometrical effects of the viscoplastic constitutive parameters on the creep curves. The influence of the viscoplastic parameters $n$ and $\omega_{vp}$ is not discussed, because these parameters do not control the shape of the curves, but only the dependency from the state of stress and the ratio between volumetric and deviatoric strain.

**Parameter $m$**

The inverse of parameter $m$ controls the slope of the inclined asymptote (Equation (6.35)) of the viscoplastic strain rate-time curve in a logarithmic diagram. Therefore an increase of $m$ lead to a decrease of this slope as represented in Figure 6.8.

\[ \text{Axial strain rate [day$^{-1}$]} \]

\[ \text{Time [day]} \]

**Figure 6.8:** Geometrical effect of parameter $m$ on the curve $\log(\varepsilon_{vp}^a) - \log(t)$ of creep
The effect of the variation of parameter $m$ on the normalized viscoplastic strain-time curve is shown in Figure 6.9. An increase of $m$ reduces the curvature of the creep curve. The limit case $m \to 0$ represents a bilinear behaviour, while the limit case $m \to \infty$ represents a linear behaviour.

![Figure 6.9: Geometrical effect of parameter $m$ on the creep curve $\varepsilon_a^v - t$](image)

**Parameter $\gamma$**

An increase of the parameter $\gamma$ leads to an upwards translation of the whole viscoplastic strain rate-time curve in a logarithmic diagram, as depicted in Figure 6.10.

**Parameter $l$**

An increase of the parameter $l$ leads to stretching towards left the whole viscoplastic strain rate-time curve in a logarithmic diagram, as depicted in Figure 6.11. The effect of $l$ is to change the time scale.
6.5. Analytical solution for creep

Figure 6.10: Geometrical effect of parameter $\gamma$ on the creep curve $\log(\dot{\varepsilon}_a^{vp}) - \log(t)$

Figure 6.11: Geometrical effect of parameter $l$ on the creep curve $\log(\dot{\varepsilon}_a^{vp}) - \log(t)$
6.6 Solution for constant strain rate

An analytical closed form solution of the SHELVIP model for constant strain rate cannot be found. However, in order to highlight the ability of the model to describe the time dependent behaviour of geomaterials in such a condition, the results of numerical solutions can be considered. To this end, let us take a triaxial constant strain rate test. Assume that, following initial loading to a stress state represented by a point in the $q-p$ plane between the viscoplastic and the plastic yield surfaces, the axial stress $\sigma_a$ is increased under a constant axial strain rate $\dot{\varepsilon}_a$ as the confining pressure $\sigma_r$ is kept constant.

For a final stress state below the plastic yield surface, the relationship between the axial strain rate and the axial and radial stresses can be expressed, by using Equations (6.2), (6.12), (6.13) and (6.22), as:

$$\dot{\varepsilon}_a = \frac{\dot{\sigma}_a}{E} + \gamma \cdot \left[ \sigma_a - \sigma_r - \alpha_{vp} \cdot \left( \frac{1}{3} \sigma_a + \frac{2}{3} \sigma_r + \frac{k_p}{\alpha_p} \right) \right]^n \cdot \left( 1 - \frac{\omega_{vp}}{3} \right)$$  \hspace{1cm} (6.38)

which can be solved numerically with Euler explicit method, if a constant and sufficiently small time step $\Delta t$ is adopted:

$$\sigma_a^{(t+\Delta t)} = \sigma_a^{(t)} + E \left\{ \dot{\varepsilon}_a + \right.$$

$$- \gamma \left[ \sigma_a^{(t)} - \sigma_r - \alpha_{vp}^{(t)} \left( \frac{1}{3} \sigma_a^{(t)} + \frac{2}{3} \sigma_r + \frac{k_p}{\alpha_p} \right) \right]^n \left( 1 - \frac{\omega_{vp}}{3} \right) \left\} \Delta t \right(6.39)$$

In a similar manner, the evolution of the viscoplastic hardening level $\alpha_{vp}$ can be calculated from Equation (6.23) as:

$$\alpha_{vp}^{(t+\Delta t)} = \alpha_{vp}^{(t)} + \frac{1}{m \cdot n} \cdot \left[ \sigma_a^{(t)} - \sigma_r - \alpha_{vp}^{(t)} \left( \frac{1}{3} \sigma_a^{(t)} + \frac{2}{3} \sigma_r + \frac{k_p}{\alpha_p} \right) \right]^{m-n+1} \cdot \left[ \sigma_a^{(t)} - \sigma_r - \alpha_{vp}^{(t)} \left( \frac{1}{3} \sigma_a^{(t)} + \frac{2}{3} \sigma_r + \frac{k_p}{\alpha_p} \right) \right]^{m-n}$$ \hspace{1cm} (6.40)

Equations (6.39) and (6.40) allow to determine the axial stress $\sigma_a^{(t)}$ for each time $t$, starting from the initial values of the axial stress $\sigma_a^{(0)}$ and the viscoplastic hardening level $\alpha_{vp}^{(0)}$. The computation can be performed by using an electronic spreadsheet.
6.6.1 Triaxial constant strain rate test

Let us consider a series of triaxial constant strain rate tests, performed under the same conditions, but characterized by different values of the axial strain rate \( \dot{\varepsilon}_a \). For each test the stress-strain curve can be determined by applying Equations (6.39) and (6.40), as shown in Figure 6.12.

Each curve exhibits a strongly non-linear behaviour below the plastic yield limit. The stiffness of the material decreases as the applied stress increases. This behaviour, which is typical of weak rocks and soils, is usually taken into account in elastoplasticity by using a non-linear elastic relationship. On the contrary, the SHELVIP model, like most of time dependent constitutive laws, can describe this behaviour as an effect of the development of viscoplastic strains.

As illustrated in Figure 6.12, in the current version of the SHELVIP model, the material is assumed to be perfectly plastic. However, a hardening or softening post-peak behaviour could be easily taken into account, by defining the dependency of the plastic yield surface from plastic strains. This can be done by changing simultaneously \( \alpha_p \) and \( k_p \), in order to maintain the ratio \( k_p/\alpha_p \) constant. This allows not to change the intersect of the plastic yield surface with the \( p \)-axis, without influencing the viscoplastic yield surface, which is defined by Equation (6.12).
Chapter 6. The SHELVIP constitutive model

According to the typical behaviour of weak rocks and soils, the stiffness increases as the applied strain rate increases. On the contrary, the peak strength is constant and does not depend on the level of the applied rate of strain. This is not supported by the results of testing, which instead highlight an increase of the peak strength with the increase of the applied strain rate (Vaid et al., 1979; Tavenas et al., 1978; Zhu et al., 1999). This behaviour is essentially due to the fact that, in the current version of the SHELVIP model, the plastic yield surface is fixed and cannot change position neither by means of the viscoplastic strains, nor by means of time. This limit will be removed in future improvements of the model, by introducing a viscoplastic hardening of the plastic yield surface.

If the axial strain rate tends to infinity (i.e. it is sufficiently large) the behaviour of the material is purely linear elastic as illustrated in Figure 6.12. According to the assumptions of the SHELVIP model, the linearity or the non-linearity is not an intrinsic property of the material but depends on the level of the applied strain rate.

Consider now a stress point located exactly on the viscoplastic yield surface. If the axial strain rate tends to zero, the increase of the axial stress tends to zero, and, as a consequence, the stress-strain curve tends to be horizontal. This is not in agreement with the existence of a “static stress-strain curve” as postulated by some authors (Vaid and Campanella, 1977; Sulem, 1983)

Finally, it is of interest to analyse the effect of a change of the applied axial

![Figure 6.13: Effect of the change of the applied rate of strain on the stress strain curve](image)

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strain rate during a constant strain rate test. As depicted in Figure 6.13, if the axial strain rate is changed from a value to another one, the stress-strain curve gradually moves according to the “isotach behaviour”, so that a unique relationship exists between stress, strain and strain rate.

### 6.6.2 Triaxial stress relaxation test

A triaxial stress relaxation test can be considered equivalent to a constant strain rate test with the applied axial strain rate $\dot{\varepsilon}_a$ equal to zero. If the starting stress point is located between the viscoplastic and the plastic yield surfaces, Equations (6.39) and (6.40) can be used to describe a stress relaxation process. Figure 6.14 shows a typical example of the curve which is obtained.

![Figure 6.14: Axial stress decrease for a triaxial relaxation test](image)

It is noted that the SHELVIP model can reproduce correctly the stress relaxation behaviour, which is widely described in literature (Vialov and Skibitsky, 1961; Sheahan et al., 1994; Silvestri et al., 1988; Zhu et al., 1999). The initial stress rate tends to infinity. Then, the stress rate decreases and tends asymptotically to a value equal to zero. A final relaxed state is therefore reached asymptotically after a certain time.

The existence of a final relaxed stress state, which is based on the experimental work carried out by a number of authors (Vialov and Skibitsky, 1961; Sheahan et al., 1994; Silvestri et al., 1988; Zhu et al., 1999), is an important
feature of the SHELVIP model which has been developed with the intent to catch this particular aspect of the time dependent response.

Finally, if the stress decrease is plotted versus the logarithm of time (Figure 6.15), a nearly linear trend can be observed at the intermediate stress level, which confirms the experimental observations reported by Lacerda and Houston (1973).

![Axial stress decrease versus logarithm of time for a triaxial stress relaxation test](image)

**Figure 6.15**: Axial stress decrease versus logarithm of time for a triaxial stress relaxation test

### 6.6.3 Apparent preconsolidation effect

The effects of ageing are very important for soils and weak rocks, and can be observed in the stress-strain relation subsequent to long periods of creep. The SHELVIP model can reproduce correctly only the accumulated effect which has been defined “ageing without structuration” (Tatsuoka et al., 2000) or “apparent preconsolidation effects”. Obviously it cannot take into account the accumulated effects related to a temporary or persistent structuration of the material.

As shown in Figure 6.16, if a material is reloaded after a creep phase, the apparent preconsolidation effect appears. The reloading stress-strain curve starts with a higher stiffness, and, after some time, it reaches again the original compression curve.
6.7 Numerical implementation

The SHELVIP constitutive model has been implemented in the FLAC code (ITASCA, 2006), a well known finite difference code, using a DLL library written with the C++ language. Before describing such a numerical implementation procedure, a brief description of the FLAC code is presented.

6.7.1 The FLAC code

The FLAC code (Fast Lagrangian Analysis of Continua) is an explicit, finite difference program, that allows to perform numerical analyses of geotechnical problems, with particular attention to geometrical and physical non-linearity.

Finite Difference Method

The finite difference method is the oldest numerical technique used for the solution of sets of differential equations, given initial values and/or boundary values. In the finite difference method, every derivative in the set of governing equations is replaced directly by an algebraic expression written in terms of the field variables (e.g., stress or displacement) at discrete points in space; these variables are undefined within elements.
Most of the finite difference codes for the solution of two-dimensional problems are limited to a rectangular discretization grid. The solution method implemented into the FLAC code is based on the Wilkins method (Wilkins, 1964), that allows to formulate the finite difference equations with reference to quadrangular elements of any shape. Therefore from this point of view the FLAC code is as performant as the finite elements method.

The finite difference grid is composed of quadrangular elements, each one divided into two couple of triangular elements (a,b and c,d in Figure 6.17). The force and displacement on each node is assumed to be equal to the average of the nodal forces and displacements on each element. This allows to obtain a symmetrical response under symmetrical boundary conditions.

![Figure 6.17: Overlaid quadrilateral elements used in FLAC](image)

The FLAC code uses Lagrangian elements, with a geometry which is updated for each computational step. This allows to deal with large deformation problems without an additional procedure. Moreover, the FLAC code makes use of an explicit solutions algorithm, without the need to form a stiffness matrix as done with the finite element method. This allows to save a large quantity of Random Access Memory (RAM). Actually only the variables of the previous computational step are saved in memory and not the entire stiffness matrix.

**Explicit solution method**

This section describes the explicit solution method which is implemented in the FLAC code from a conceptual point of view. The method is presented first with reference to the elastoplastic behaviour and then it is extended to the time dependent behaviour. The numerical procedures for the implementation are not discussed in this section. Further information can be found in Marti and Cundall (1982).

The explicit solution method is inspired to the principle of propagation and dissipation of the kinetic energy into a deformable body in movement.
The solution algorithm considers this physical phenomenon by means of the equations of the dynamic of the motion.

The aim of the Lagrangian explicit solution method is to resolve a static problem (elastoplastic) or quasi-static problem (viscoplastic) by means of the equations of motion. The solution scheme is depicted in Figure 6.18, with reference to a computational step $\Delta t$.

![Figure 6.18: Basic explicit calculation cycle used in FLAC](image)

The solution process starts with the equations of motion, that provide a new field of nodal velocities by means of the integration of the accelerations on the time $\Delta t/2$, as illustrated in Figure 6.19. A second integration on the time $\Delta t/2$ allows to find the field of nodal displacements.

The equations of motion of Newton for a deformable body in a Lagrangian reference system can be expressed as:

$$
\rho_v \cdot \frac{\partial \dot{u}_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho_v \cdot g_i
$$

(6.41)

where $\rho_v$ is the volumetric weight, $\dot{u}_i$ the term of nodal velocity, $x_j$ the term of spacial position, $g_i$ the term of volumetric acceleration vector, $\sigma_{ij}$ the stress tensor, and $t$ the time.

The strain tensor of the material during the time $\Delta t$ can be written as:

$$
\Delta \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} \right) \cdot \Delta t
$$

(6.42)

The new stress tensor is then evaluated by means of the constitutive law of the material, which implementation is described in the following section. The new nodal forces, that can be calculated from the stress field, allow to evaluate the new field of acceleration at the end of the computational step $\Delta t$. 
In each box in the scheme of Figure 6.18 each quantity is increased from the value of the previous computational step.

The determination of the new stresses does not influence the nodal velocities evaluated in the previous box. This hypothesis is true if the computational time step $\Delta t$ is chosen small enough, that the disequilibrium generated inside an element cannot propagate to the neighbouring elements during the same time interval.

The solution procedure is not unconditionally stable. A critical time step $\Delta t_c$, that cannot be exceeded by the computational time step $\Delta t$, must be defined. Cormeau (1975) and Billaux and Cundall (1993) stated that the evaluation of the critical time step can be done upon the principle that states that the velocity of the solution wave must be always greater than the velocity of the physical wave during a computational cycle. The procedure for the evaluation of the critical time step will be discussed in the following.

The system is highly unstable during the first computational steps because of the high unbalanced forces. In order to achieve a stable solution algorithm the movements of the solid must be damped. This allows to reach, as fast as possible, a residual state of disequilibrium which is negligible comparing to the initial state of stress. The damping algorithms implemented in FLAC are described in the following section.

The stability criterion, which allows to control the state of equilibrium of
the system, is based on the maximum unbalanced force. The user defines a value of force, below that the equilibrium is supposed to be achieved.

The explicit solution method implemented in FLAC leads to a high number of computational steps, that makes this method slower than the finite element method for linear elastic problems.

**Mechanical damping**

The mechanical damping of the system is based on the reduction of the residual forces (unbalanced forces) and on the nodal velocities. It consists in applying an additional force to each node, with modulus proportional to the non-equilibrated force of the previous step. The directions of these forces are such as to make a dissipative work. The damping allows to dissipate the unbalanced forces at equilibrium.

The discretization of the equation of the motion for an element \( E_j \) leads to:

\[
\dot{u}_i^{(t+\Delta t/2)} = \dot{u}_i^{(t-\Delta t/2)} + \frac{1}{\rho_v} \left\{ \sum_{E_j} F^{(t)}_i - F^{(t)}_{di} \right\} \Delta t
\]

(6.43)

where \( \rho_v \) is the volumetric weight, \( F^{(t)}_{di} \) is the non-equilibrated force for the element \( E_j \) at time \( t \), and \( \sum_{E_j} F^{(t)}_i \) is the vectorial sum of the nodal forces of element \( E_j \) at time \( t \).

In the FLAC code the default damping method is called “local damping”. The oscillation of the computational wave is damped by considering the change of the sign of the nodal velocities. The expression of the non-equilibrated force is:

\[
F^{(t)}_{di} = \alpha_d \cdot \left| \sum_{E_j} F^{(t)}_i \right| \times \text{sign} \left( \dot{u}_i^{(t-\Delta t/2)} \right)
\]

(6.44)

where \( \alpha_d \) is a damping constant with a default value equal to 0.8. The damping constant does not depend on the properties of the system, therefore it acts with different efficiency on each element of the grid.

This type of damping can be inefficient in some cases. A significant case is viscoplasticity, where the nodal velocity is highly conditioned by the viscoplastic strain rate (e.g., secondary phase of creep). The local damping in this case can prevent the correct development of the nodal velocities, and can generate some localized zones with high numerical instability. Therefore, for viscoplastic analysis it is suitable to use a second damping algorithm called “combined damping”.

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The combined damping is based both on the change of sign of the unbalanced forces and on the change of sign of the nodal velocities. Therefore the non-equilibrated force can be written as:

\[
F_{di}^{(t)} = \alpha_d \sum_{E_j} F_{i}^{(t)} \times \frac{1}{2} \left\{ \text{sign} \left( F_{i}^{(t)} - F_{i}^{(t-\Delta t)} \right) - \text{sign} \left( \dot{u}_{i}^{(t-\Delta t/2)} \right) \right\} \quad (6.45)
\]

Practically the local damping can be used only if the viscoplastic deformations are concentrated in localized zones and do not affect the whole model (e.g. is the case of tunnel excavation).

**Critical time step**

As illustrated above, the solution procedure of the FLAC code is not unconditionally stable and a critical time step \( \Delta t_c \) must be defined. Marti and Cundall (1982) stated that for an elastic solid discretized using a squared grid, with edge of \( \Delta x \), the stability condition can be expressed as:

\[
\Delta t < \Delta t_c = \frac{\Delta x}{C_p} \quad (6.46)
\]

where \( \Delta t \) is the computational time step, \( \Delta t_c \) the critical time step, and \( C_p \) the velocity of the computational wave.

For an elastic medium \( C_p \) can be assumed equal to the velocity of the primary p-waves, which is defined as follows:

\[
C_p = \sqrt{\frac{K + \frac{4}{3}G}{\rho_v}} = \sqrt{\frac{E_{edo}}{\rho_v}} \quad (6.47)
\]

where \( K, G, E_{edo} \) are the bulk modulus, the shear modulus, and the oedometer modulus respectively.

In viscoplasticity the computational time corresponds to the physical time and the explicit solution must not introduce high unbalanced forces during the first steps. Therefore the numerical solution must follow as accurately as possible the physical evolution of the real problem.

For any viscoplastic law, that states that the viscoplastic deformations are controlled only by the deviatoric state of stress, it is suggested that the critical time step can be evaluated as the ratio between the viscosity of the material \( \eta \) and its shear modulus \( G \) (ITASCA, 2006) as:

\[
\Delta t_c = \frac{\eta}{G} \quad (6.48)
\]
Writing new constitutive models

The FLAC code allows the user to write new constitutive models. This can be done by using a special internal programming language called FISH (FlacISH), or by means of a DLL library written with the C++ language. The first method is more simple, because of the simplicity of the scripting language, but computationally it is much more slow, because the script code is not compiled.

The procedure consists of the following steps:

- Initialization of the variables. This procedure consists to set the constitutive parameters and the global variable of the law. It is executed only once for each element and for each solution.

- Determination of the new stress. This procedure is the core of the implementation, and consists in evaluating the new state of stress starting from the strain increments $\Delta \varepsilon_{ij}$ and from the internal hardening parameters. It is called four times for each element (one for each sub zone) and for time step.

- Evaluation of the maximum modulus. This procedure consists in defining the maximum shear modulus and the maximum oedometer modulus in order to permit the FLAC code to evaluate the admissible time step.

- Thermomechanics coupling. This procedure is executed only during thermal analysis and permits the thermomechanical coupling.

The constitutive law is written for each sub-zone (with a number of 2 or 4 depending on the geometry of the element). Therefore the state parameters and the hardening parameters are determined as the averages of these quantities for all the sub-zones, by considering the area or the volume of each sub-zone.

6.7.2 Implementation

In this section the numerical implementation of the SHELVIP constitutive model into the FLAC code is described in detail. It has been carried out with reference to full three-dimensional conditions in order to allow the code to be used for both two-dimensional analysis with FLAC$^{2D}$ and for three-dimensional analysis with FLAC$^{3D}$.

A new DLL library has been written, optimized, and compiled using the C++ language in order to obtain the maximum computational speed as possible. It is in fact very important to remember that for complex analyses and for back analyses the required time is extremely important and must be reduced to a minimum.
Figure 6.20: Conceptual scheme of the implementation procedure of the SHELVIP model in FLAC
6.7. Numerical implementation

The conceptual scheme of the numerical implementation is illustrated in Figure 6.20.

The constitutive law can be represented in FLAC like a black box, which is characterized by input variables and output variables.

The input variables for the SHELVIP model are:

- The tensor of the increments of the total strains $\Delta \varepsilon_{ij}$. It is determined by the solver for each computational step by means of the equation of motion and by means of the stress state $\sigma^{O}_{ij}$, which has been evaluated using the constitutive law in the previous step.

- The tensor of the stresses $\sigma^{O}_{ij}$ which has been evaluated using the constitutive law in the previous step.

- The viscoplastic hardening level $\alpha^{O}_{vp}$ which has been evaluated using the constitutive law in the previous step. If the current step is the first one the initial viscoplastic hardening level $\alpha_{vp,0}$ is used.

The output variables are:

- The new tensor of the stresses $\sigma^{N}_{ij}$. It will be used in the next step to solve the equation of movement and in the constitutive law.

- The new viscoplastic hardening level $\alpha^{N}_{vp}$. It will be used in the next step in the constitutive law.

The numerical implementation of the SHELVIP model can be subdivided into three principal blocks:

- evaluation of the first trial elastic-viscoplastic stresses;
- evaluation of plastic corrections;
- update of the viscoplastic hardening level.

These blocks are described with accuracy in the following.

It is important to remember that the FLAC sign convention takes tensile stress as positive.

**Evaluation of the first trial elastic-viscoplastic stresses**

The first block consists in determining the elastic-viscoplastic stresses of first trial $\sigma^{I}_{ij}$.

First of all the volumetric stress $p^{O}$ and the deviatoric stress $q^{O}$ are determined with reference to the old state of stress $\sigma^{O}_{ij}$, which has been calculated in the previous step:
Chapter 6. The SHELVIP constitutive model

\[ p^O = \frac{1}{3} \sigma^{O}_{mm} \]  \quad (6.49)

\[ s^O_{ij} = \sigma^O_{ij} - \delta_{ij} \cdot p^O \]  \quad (6.50)

\[ q^O = \sqrt{\frac{3}{2}} s^O_{ij} \cdot s^O_{ij} \]  \quad (6.51)

At this point it is necessary to evaluate if the old state of stress \((p^O, q^O)\) exceeds or not the viscoplastic yield surface, which is defined by the old viscoplastic hardening level \(\alpha^O_{vp}\).

The \(q - p\) stress plane is divided in elastic and viscoplastic fields, by the viscoplastic yield function \(f_{vp}\) and by the tensile strength \(\sigma_t\) as depicted in Figure 6.21.

![Figure 6.21: Elastic and viscoplastic stress fields for the implementation of the SHELVIP model](image)

It is necessary to evaluate the sign of the two following functions:

\[ f^O_{vp} = q^O - \alpha^O_{vp} \cdot \left( -p^O + \frac{k_p}{\alpha_p} \right) \]  \quad (6.52)

\[ h^O_{vp} = p^O - \sigma_t \]  \quad (6.53)
If the old stress state \((p^O, q^O)\) is inside the elastic field \((f^O_{vp} < 0 \ OR \ h^O_{vp} > 0)\) the increment of the elastic strains \(\Delta \varepsilon_{ij}^e\) is equal to the increment of the total strain \(\Delta \varepsilon_{ij}\):

\[
\Delta \varepsilon_{ij}^e = \Delta \varepsilon_{ij} \tag{6.54}
\]

If the old stress state \((p^O, q^O)\) is inside the viscoplastic field \((f^O_{vp} > 0 \ AND \ h^O_{vp} < 0)\), the viscoplastic strain rates \(\dot{\varepsilon}_{ij}^{vp}\) are evaluated using Equation (6.22):

\[
\dot{\varepsilon}_{ij}^{vp} = \gamma \cdot (f^O_{vp})^n \cdot \left( \frac{3}{2} \cdot \frac{s^O_{ij}}{q^O} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right) \tag{6.55}
\]

The increment of the viscoplastic strain \(\Delta \varepsilon_{ij}^{vp}\) over the time \(\Delta t\) is evaluated as:

\[
\Delta \varepsilon_{ij}^{vp} = \dot{\varepsilon}_{ij}^{vp} \cdot \Delta t \tag{6.56}
\]

The elastic strain increments \(\Delta \varepsilon_{ij}^e\) are then evaluated by subtracting to the total strain increments \(\Delta \varepsilon_{ij}\) the viscoplastic strain increments \(\Delta \varepsilon_{ij}^{vp}\):

\[
\Delta \varepsilon_{ij}^e = \Delta \varepsilon_{ij} - \Delta \varepsilon_{ij}^{vp} \tag{6.57}
\]

Finally the elastic-viscoplastic stresses of first trial \(\sigma_{ij}^I\) are evaluated from the old stresses \(\sigma_{ij}^O\) by using the elastic strain increment \(\Delta \varepsilon_{ij}^e\) and the stiffness matrix \(E_{ijkl}\):

\[
\sigma_{ij}^I = \sigma_{ij}^O + E_{ijkl} \cdot \Delta \varepsilon_{kl}^e \tag{6.58}
\]

### Evaluation of plastic corrections

The second block consists in evaluating the new stress state \(\sigma_{ij}^N\), by means of the plastic corrections determined from the elastic-viscoplastic stresses of first trial \(\sigma_{ij}^I\).

First of all, the volumetric stress \(p^I\) and the deviatoric stress \(q^I\) are determined with reference to the first trial stresses \(\sigma_{ij}^I\):

\[
p^I = \frac{1}{3} \sigma_{mm}^I \tag{6.59}
\]

\[
s_{ij}^I = \sigma_{ij}^I - \delta_{ij} \cdot p^I \tag{6.60}
\]

\[
q^I = \sqrt{\frac{3}{2} s_{ij}^I \cdot s_{ij}^I} \tag{6.61}
\]
At this point it is necessary to evaluate if the first trial state of stress \((p^I, q^I)\) exceeds or not the plastic yield surface.

![Figure 6.22: Plastic stress fields for the implementation of the SHELVIP model](image)

As depicted in Figure 6.22, the \(q - p\) plane is divided into three fields by means of the plastic yield surface \(f_p\), the tension strength \(h_p\), and the line \(r_p\) which is the bisector of the two previous lines:

\[
f_p^I = q^I + \alpha_p \cdot p^I - k_p \tag{6.62}
\]

\[
h_p^I = p^I - \sigma_t \tag{6.63}
\]

\[
r_p^I = q^I - \beta_p \cdot p^I - b_p \tag{6.64}
\]

where:

\[
\beta_p = \sqrt{\alpha_p^2 + 1} - \alpha_p \tag{6.65}
\]

\[
b_p = k_p - \sqrt{\alpha_p^2 + 1} \tag{6.66}
\]

The stress fields are:

- a non-plastic field \((f_p^I < 0 \text{ AND } h_p^I < 0)\)
6.7. Numerical implementation

- a plastic shear field \((f^I_p > 0 \text{ AND } r^I_p > 0)\)
- a plastic tension field \((r^I_p < 0 \text{ AND } h^I_p > 0)\)

If the first trial stress \((q^I, p^I)\) is inside the non-plastic field no plastic deformations develop and the new state of stress \(\sigma^N_{ij}\) is equal to the first trial state of stress \(\sigma^I_{ij}\):

\[
\sigma^N_{ij} = \sigma^I_{ij}
\] (6.67)

If the first trial stress \((q^I, p^I)\) is inside the plastic shear field it is necessary to evaluate the plastic correction with the following procedure.

The plastic strain increments \(\Delta \varepsilon^p_{ij}\) can be determined by using the plastic flow rule of Equation (6.15):

\[
\Delta \varepsilon^p_{ij} = \lambda^s \cdot \frac{\partial g^s_p}{\partial \sigma_{ij}}
\] (6.68)

where \(\lambda^s\) is the plastic multiplier for shear and \(g^s_p\) the plastic potential for shear.

The volumetric plastic strain increments \(\Delta \varepsilon^p_v\) and the deviatoric plastic strain increments \(\Delta \varepsilon^p_q\) can be determined as:

\[
\begin{aligned}
\Delta \varepsilon^p_v &= \lambda^s \cdot \frac{\partial g^s_p}{\partial p} \\
\Delta \varepsilon^p_q &= \lambda^s \cdot \frac{\partial g^s_p}{\partial q}
\end{aligned}
\] (6.69)

The expression of the plastic potential for shear \(g^s_p\) is (Equation (6.16)):

\[
g^s_p = q + \omega_p \cdot p
\] (6.70)

Therefore the Equations (6.69) can be rewritten as:

\[
\begin{aligned}
\Delta \varepsilon^p_v &= \lambda^s \cdot \omega_p \\
\Delta \varepsilon^p_q &= \lambda^s
\end{aligned}
\] (6.71)

The new volumetric stress \(p^N\) and the new deviatoric stress \(q^N\) can be evaluated from the old values as:

\[
\begin{aligned}
p^N &= p^O + K \cdot (\Delta \varepsilon^p_v - \Delta \varepsilon^v_p - \Delta \varepsilon^p) \\
q^N &= q^O + G \cdot (\Delta \varepsilon^p_q - \Delta \varepsilon^q_v - \Delta \varepsilon^p_q)
\end{aligned}
\] (6.72)

where \(K\) and \(G\) are the elastic bulk modulus and the elastic shear modulus respectively.
By remembering the definition of the first trial stress (Equation (6.58)), it is possible to write:

\[
\begin{align*}
    p^N &= p^I - K \cdot \lambda^s \cdot \omega_p \\
    q^N &= q^I - G \cdot \lambda^s 
\end{align*}
\]  

(6.73)

The value of the plastic multiplier for shear \( \lambda^s \) is determined by using the consistency condition. The new state of stress \((q^N, p^N)\) must be on the plastic yield surface \( f^N_p = 0 \):

\[
f^N_p = q^N + \alpha_p \cdot p^N - k_p = q^I - G \cdot \lambda^s + \alpha_p \cdot (p^I - K \cdot \lambda^s \omega_p) - k_p = 0
\]

(6.74)

That can be solved for \( \lambda^s \):

\[
\lambda^s = \frac{q^I + \alpha_p \cdot p^I - k_p}{G + K \cdot \alpha_p \cdot \omega_p} = \frac{f^I_p}{G + K \cdot \alpha_p \cdot \omega_p}
\]

(6.75)

Therefore new stress \( p^N \) and \( q^N \) can be evaluated from Equation (6.73) as:

\[
\begin{align*}
    p^N &= p^I - \frac{f^I_p}{G + K \cdot \alpha_p \cdot \omega_p} \cdot K \cdot \omega_p \\
    q^N &= q^I - \frac{f^I_p}{G + K \cdot \alpha_p \cdot \omega_p} \cdot G
\end{align*}
\]

(6.76)

Noting that the new deviatoric stresses \( s^N_{ij} \) may be obtained by multiplying the corresponding deviatoric elastic-viscoplastic guesses \( s^I_{ij} \) with the ratio \( q^N / q^I \), the new stresses \( \sigma^N_{ij} \) may be written:

\[
\sigma^N_{ij} = s^I_{ij} \cdot \frac{q^N}{q^I} + \delta_{ij} \cdot p^N
\]

(6.77)

If the first trial stress \((q^I, p^I)\) is inside the plastic tensile field it is necessary to evaluate the plastic correction with the following procedure.

The plastic strain increment \( \Delta \varepsilon^p_{ij} \) can be determined by using the plastic flow rule of Equation (6.15):

\[
\Delta \varepsilon^p_{ij} = \lambda^t \cdot \frac{\partial g^t_p}{\partial \sigma_{ij}}
\]

(6.78)

where \( \lambda^t \) is the plastic tensile multiplier and \( g^t_p \) the plastic tensile potential.

The volumetric plastic strain increment \( \Delta \varepsilon^p_q \) and the deviatoric plastic strain increment \( \Delta \varepsilon^p_q \) can be determined as:
6.7. Numerical implementation

\[
\begin{align*}
\Delta \varepsilon_p^p &= \lambda^t \cdot \frac{\partial g_p^t}{\partial p} \\
\Delta \varepsilon_q^p &= \lambda^t \cdot \frac{\partial g_p^t}{\partial q}
\end{align*}
\]  
(6.79)

The expression of the plastic potential for shear \( g_p^t \) is:

\[ g_p^t = p \]  
(6.80)

Therefore the Equations (6.79) can be rewritten as:

\[
\begin{align*}
\Delta \varepsilon_p^p &= \lambda^t \\
\Delta \varepsilon_q^p &= 0
\end{align*}
\]  
(6.81)

Applying a reasoning similar to that described above, it is possible to obtain:

\[
\begin{align*}
p^N &= p^I - K \cdot \lambda^t \\
q^N &= q^I
\end{align*}
\]  
(6.82)

and applying the consistency condition:

\[ \lambda^t = \frac{p^I - \sigma_t}{K} \]  
(6.83)

As expected, substitution of this expression in Equation (6.82) yields:

\[
\begin{align*}
p^N &= \sigma_t \\
q^N &= q^I
\end{align*}
\]  
(6.84)

In this mode of failure, the new deviatoric stresses \( q^N \) correspond to the elastic-viscoplastic guess \( q^I \) and we may write:

\[ \sigma_{ij}^N = \sigma_{ij}^I + (\sigma_t - p^I) \cdot \delta_{ij} \]  
(6.85)

**Update viscoplastic hardening level**

The third block consists in evaluating the new viscoplastic hardening level \( \alpha_{vp}^N \) if the new state of stress \( \sigma_{ij}^N \) exceeds the viscoplastic yield surface evaluated with reference to the old viscoplastic hardening level \( \alpha_{vp}^O \).

First of all the new volumetric stress \( p^N \) and the new deviatoric stress \( q^N \) are determined with reference to the new state of stress \( \sigma_{ij}^N \):

\[ p^N = \frac{1}{3} \cdot \sigma_{mm}^N \]  
(6.86)
\[ s_{ij}^N = \sigma_{ij}^N - \delta_{ij} \cdot p^N \] (6.87)

\[ q^N = \sqrt{\frac{3}{2} s_{ij}^N \cdot s_{ij}^N} \] (6.88)

At this point it is necessary to evaluate if the new state of stress \((p^N, q^N)\) exceeds or not the viscoplastic yield surface, which is defined by the old viscoplastic hardening level \(\alpha_{vp}^O\). The procedure is similar to the one of the first block.

If the new state of stress \(\sigma_{ij}^N\) does not exceed the viscoplastic yield surface \((f_{vp}^N < 0 \text{ OR } h_{vp}^N > 0)\) the new viscoplastic hardening level \(\alpha_{vp}^N\) is equal to the old value \(\alpha_{vp}^O\):

\[ \alpha_{vp}^N = \alpha_{vp}^O \] (6.89)

If the new state of stress \(\sigma_{ij}^N\) exceeds the viscoplastic yield surface \((f_{vp}^N > 0 \text{ AND } h_{vp}^N < 0)\) the rate of the viscoplastic hardening is calculated using the Equation (6.23):

\[ \dot{\alpha}_{vp} = \frac{l}{m \cdot n} \cdot \frac{f_{vp}^N}{p^N + \frac{k_p}{\alpha_p}} \cdot \left( \frac{f_{vp}^N}{q^N} \right)^{m \cdot n} \] (6.90)

Finally the new hardening level \(\alpha_{vp}^N\) is determined by adding to the old value \(\alpha_{vp}^O\) the integration of the hardening rate \(\dot{\alpha}_{vp}\) over the time interval \(\Delta t\):

\[ \alpha_{vp}^N = \alpha_{vp}^O + \dot{\alpha}_{vp} \cdot \Delta t \] (6.91)

### 6.7.3 Validation

This section describes the validation of the numerical implementation of the SHELVIP constitutive model into the FLAC code by means of some examples.

The example considered in this thesis are: (1) a multi-stage triaxial strength test, (2) a triaxial creep test, (3) a triaxial stress relaxation test. The first test is used to verify the elastic-plastic behaviour of the SHELVIP model, while the last two tests are used to verify the viscoplastic behaviour.

The validation is performed by comparing the results of numerical analysis with the results of analytical closed-form solutions or incremental solutions, when an analytical solution is not available.

All the numerical analyses have been performed with the FLAC\textsuperscript{2D} code in axisymmetric conditions, with reference to a quarter of a cylindrical specimen.
6.7. Numerical implementation

Figure 6.23: Numerical model of the specimen used to validate the implementation of the SHELVIP model into the FLAC code

Table 6.1: Parameters of the SHELVIP model calibrated on the creep tests on clay shales and used in validation tests (time in day and pressure in kPa)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus $E$</td>
<td>50000 kPa</td>
</tr>
<tr>
<td>Poisson’s coefficient $\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td>Strength slope $\alpha_p$</td>
<td>0.607</td>
</tr>
<tr>
<td>Strength intercept $k_p$</td>
<td>42.34 kPa</td>
</tr>
<tr>
<td>Tensile strength $\sigma_t$</td>
<td>-10.00 kPa</td>
</tr>
<tr>
<td>Plastic dilatancy $\omega_p$</td>
<td>0</td>
</tr>
<tr>
<td>Fluidity parameter $\gamma$</td>
<td>2.264E-5</td>
</tr>
<tr>
<td>Shape factor $m$</td>
<td>1.005</td>
</tr>
<tr>
<td>Load dependency $n$</td>
<td>1.448</td>
</tr>
<tr>
<td>Time stretching $l$</td>
<td>144.86</td>
</tr>
<tr>
<td>Viscoplastic dilatancy $\omega_{vp}$</td>
<td>-0.41</td>
</tr>
</tbody>
</table>
with diameter of 50 mm and height of 100 mm (Figure 6.23). The finite difference mesh used is composed by 5 x 10 squared elements with edges of 5 mm. The boundary conditions are: (1) fixed y-displacement on the bottom side, (2) fixed x-displacement on the left side, (3) stress $\sigma_a$ or constant strain rate $\dot{\epsilon}_a$ applied on the top side, (4) stress $\sigma_r$ applied on the left side. All the analyses has been performed in dried conditions.

The constitutive parameters of the SHELVIP model have been determined with reference to the results of triaxial tests performed on clay shales (argille scagliece) by Bonini (2003). The fitting procedure is described in Section 6.8. Table 6.1 shows the values of the constitutive parameters.

**Triaxial multi-stage strength test**

This test has been performed in order to verify the elastic-plastic behaviour of the implementation of the SHELVIP model into the FLAC code.

Four compression stages have been performed at different confining pressures $\sigma_r = 0$ kPa, 100 kPa, 200 kPa and 300 kPa, by applying a constant axial strain rate $\dot{\epsilon}_a$, until the yielding of the material. After yielding the confining pressure $p$ has been increased isotropically while the deviatoric stress $q$ has been held constant. This process has been repeated four times.

The numerical stress-path is shown in Figure 6.24. It is possible to observe that the strength points obtained by the numerical analysis are exactly on the theoretical plastic yield surface.

Figure 6.25 shows the deviatoric stress $p$ plotted versus the deviatoric strain $\varepsilon_q$ and the volumetric strain $\varepsilon_p$. It can be observed that the elastic segments of the two curves match exactly the theoretical elastic stress-strain lines that can be expressed as:

$$\begin{align*}
\varepsilon_q^e &= \frac{2(1 + \nu)}{E} \cdot q \\
\varepsilon_p^e &= \frac{1 - 2 \cdot \nu}{E} \cdot q
\end{align*}$$

(6.92)

The part of the $q - \varepsilon_q$ curve immediately following the yielding is horizontal because the plastic yield surface cannot harden (i.e. perfect plasticity).

If the deviatoric strain $\varepsilon_q$ is plotted versus the volumetric strain $\varepsilon_p$ (Figure 6.26) it is possible to observe:

- an elastic inclined compression phase; the deviatoric and the volumetric strains follow exactly the theoretical elastic behaviour which can be expressed by the relationship:

$$\varepsilon_q^e = \frac{2(1 + \nu)}{1 - 2 \nu} \cdot \varepsilon_p^e$$

(6.93)
6.7. Numerical implementation

Figure 6.24: Stress-path for triaxial multi-stage strength validation test

Figure 6.25: Stress-strain curves for triaxial multi-stage strength validation test
• a plastic vertical phase; only the deviatoric strain $\varepsilon_q$ develops during this phase because a plastic dilatancy equal to zero $\omega_p = 0$ has been assumed;

• an elastic horizontal consolidation phase; only the volumetric strain $\varepsilon_p$ develops during this phase because the load is incremented isotropically.

Figure 6.26: Deviatoric strains versus volumetric strains for triaxial multi-stage strength test

Therefore it is possible to state that the implementation of the elastic and plastic component of the SHELVIP model in the FLAC code is valid.

Triaxial creep test

This test has been performed in order to verify the implementation of the viscoplastic component of the SHELVIP model into the FLAC code.

The sample has been loaded elastically up to the deviatoric stress $q=271$ kPa ($\sigma_a=1031$ kPa, $\sigma_r=760$ kPa). Then the state of stress has been kept constant and the deformations of the material have been monitored over time. An initial viscoplastic hardening level $\alpha_{vp,0}$ equal to zero has been assumed.

Figure 6.27 shows a comparison between the results of the numerical analysis with FLAC and the results of the analytical closed form solution given by Equations (6.28) and (6.29) As can be easily seen, the correspondence is perfect.

Figure 6.28 shows the comparison with reference to the axial strain rates in a logarithmic diagram. The analytical solution for the creep strain rate is
6.7. Numerical implementation

Figure 6.27: Comparison of numerical and analytical creep curves for triaxial creep test

Figure 6.28: Comparison of numerical and analytical strain rate curves for triaxial creep test
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calculated using Equation (6.27). As can be easily seen, also in this case the correspondence is perfect. The little scattering of the numerical solution is due to numerical approximations.

Therefore it is possible to state that the implementation of the viscoplastic component of the SHELVIP model in the FLAC code is valid.

**Triaxial stress relaxation test**

This test has been carried out in order to verify the validity of the implementation of the elastic and viscoplastic components of the SHELVIP model in the FLAC code. The elastic and viscoplastic constitutive parameters are the same as those adopted with the previous creep test, and are shown in Table 6.1.

The sample has been loaded elastically up to the deviatoric stress \( q = 271 \) kPa \((\sigma_a = 1031 \) kPa, \( \sigma_r = 760 \) kPa). Then the axial displacement \( \varepsilon_a \) and the radial stress \( \sigma_r \) have been kept constant, while the variation of the axial stress \( \sigma_a \) has been measured over time. An initial viscoplastic hardening level \( \alpha_{vp,0} \) equal to zero has been assumed.

Figure 6.29 shows the comparison between the results of the numerical analysis and the results of the incremental solution given by Equations (6.39) and (6.40). As can be easily seen, the correspondence is perfect.

Therefore also this test verifies successfully the numerical implementation of the SHELVIP model in the FLAC code.

![Comparison of the results of the numerical analysis and the results of the incremental solution for stress relaxation test](image)

**Figure 6.29:** Comparison of the results of the numerical analysis and the results of the incremental solution for stress relaxation test
6.8 Analytical calibration procedure

Following the illustration of the most relevant features of behaviour which characterise the SHELVIP model and the implementation into the FLAC code, the attention is now posed on the calibration of constitutive parameters based on the results of laboratory tests. The difficulties associated with such a process usually increase with the increase of the complexity of the model. This holds true for viscoplastic models, which are generally characterized by a large number of constitutive parameters. Also, the identification process is often more difficult by the unclear physical meaning of the constitutive parameters being used.

The calibration process can be performed by means of analytical or semi-analytical fitting, if a closed form solution is available, or by means of numerical optimization fitting, if the complexity of the problem is too high. Analytical fitting allows one to handle the constitutive parameters with more awareness, while numerical fitting is generally faster and more accurate. If possible, one should find a first set of parameters by analytical fitting, subsequently to be refined by a numerical optimization procedure.

In this section the analytical calibration of the constitutive parameters of the SHELVIP model is described with reference to the results of laboratory tests performed on clay shales, a rock formation from the Raticosa tunnel, which experienced very important squeezing problems during excavation (Bonini et al., 2007).

As described in Section 6.4.7, the constitutive parameters of the SHELVIP model can be subdivided in three different groups: elastic, plastic, and viscoplastic. Because of the mathematical formulation of the model, the calibration can be performed independently for each group of parameter.

6.8.1 Elastic and plastic parameters

The elastic and plastic parameters are coincident with the elastic and plastic parameters of the classical theory of elastoplasticity, as often used in design practice. Although there is no need to describe here the calibration procedure used for them, it is important to remember that the elastic modulus should be evaluated from the unloading phase, and the plastic parameters should be determined as peak values.

The elastic and plastic parameters of clay shales are summarised in Table 6.2. The strength parameters are calculated directly from the Mohr-Coulomb parameters \(c' = 20 \text{kPa}, \phi' = 16^\circ, \sigma_t = 5 \text{kPa}\) by using a circumscribing Drucker-Prager’s criterion. The plastic dilatancy is assumed to be equal to zero, in absence of direct measurements.
Table 6.2: Parameters of the SHELVIP model calibrated on the creep tests on clay shales (time in day and pressure in kPa)

<table>
<thead>
<tr>
<th></th>
<th>Analytical fitting</th>
<th>Numerical fitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus</td>
<td>$E$ 50000 kPa</td>
<td>$E$ 50000 kPa</td>
</tr>
<tr>
<td>Poisson’s coefficient</td>
<td>$\nu$ 0.3</td>
<td>$\nu$ 0.3</td>
</tr>
<tr>
<td>Strength slope</td>
<td>$\alpha_p$ 0.607</td>
<td>$\alpha_p$ 0.607</td>
</tr>
<tr>
<td>Strength intercept</td>
<td>$k_p$ 42.34 kPa</td>
<td>$k_p$ 42.34 kPa</td>
</tr>
<tr>
<td>Tensile strength</td>
<td>$\sigma_t$ -10.00 kPa</td>
<td>$\sigma_t$ -10.00 kPa</td>
</tr>
<tr>
<td>Plastic dilatancy</td>
<td>$\omega_p$ 0</td>
<td>$\omega_p$ 0</td>
</tr>
<tr>
<td>Fluidity parameter</td>
<td>$\gamma$ 1.097E-5</td>
<td>$\gamma$ 2.264E-5</td>
</tr>
<tr>
<td>Shape factor</td>
<td>$m$ 0.928</td>
<td>$m$ 1.005</td>
</tr>
<tr>
<td>Load dependency</td>
<td>$n$ 1.411</td>
<td>$n$ 1.448</td>
</tr>
<tr>
<td>Time stretching</td>
<td>$l$ 30.31</td>
<td>$l$ 144.86</td>
</tr>
<tr>
<td>Viscoplastic dilatancy</td>
<td>$\omega_{vp}$ -0.41</td>
<td>$\omega_{vp}$ -0.41</td>
</tr>
</tbody>
</table>

6.8.2 Viscoplastic parameters

The identification of the viscoplastic parameters is complex, due to the complexity and generality of the SHELVIP model. At least two triaxial creep tests, performed at different deviatoric stress levels or, alternatively, one triaxial stress relaxation test, are required in order to assess the stress dependency. Radial displacement measurements are necessary to determine the viscoplastic dilatancy.

With the intent to calibrate the viscoplastic parameters of clay shales, three triaxial undrained creep tests (Bonini, 2003; Bonini et al., 2007) performed at different stress levels are considered (Table 6.3). The main difficulty of these tests is that undrained creep is not rigorously a “pure creep”, as the variation of pore pressure during the test can modify significantly the effective state of stress.

Table 6.3: Triaxial creep tests performed on the clay shales

<table>
<thead>
<tr>
<th>Test</th>
<th>Type</th>
<th>$B$</th>
<th>$bp$</th>
<th>$\sigma_c$</th>
<th>$\dot{\varepsilon}_a$</th>
<th>$t_{max}$</th>
<th>$s_{const}$</th>
<th>$\Delta u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RTC3</td>
<td>CIU+Ucreep</td>
<td>0.77</td>
<td>399</td>
<td>497</td>
<td>0.005</td>
<td>148</td>
<td>480</td>
<td>18</td>
</tr>
<tr>
<td>RTC4</td>
<td>CIU+Ucreep</td>
<td>0.80</td>
<td>404</td>
<td>488</td>
<td>0.005</td>
<td>94</td>
<td>453</td>
<td>31</td>
</tr>
<tr>
<td>RTC5</td>
<td>CIU+Ucreep</td>
<td>0.65</td>
<td>396</td>
<td>501</td>
<td>0.001</td>
<td>134</td>
<td>491</td>
<td>9</td>
</tr>
</tbody>
</table>

$B=$Skempton’s parameter; $bp=$back pressure; $\sigma_c=$consolidation effective stress; $\dot{\varepsilon}_a=$axial strain rate in the shearing phase; $t_{max}=(\sigma_a-\sigma_r)_{max}/2$; $s_{const}=(\sigma_a+\sigma_r)_{const}/2$; $\Delta u=$excess pore pressure
inside the specimen. It is important to observe that undrained conditions were in this case strictly required, to prevent the development of swelling deformations, that are significant in the clay shales (Bonini, 2003; Bonini et al., 2007). In order to overcome this limitation, the influence of pore pressure is not taken into account and all the creep tests are treated as drained.

The first viscoplastic constitutive parameter, which can be easily determined, is the viscoplastic dilatancy $\omega_{vp}$, which defines the ratio between the volumetric $\Delta \varepsilon_p^{vp}$ and the deviatoric $\Delta \varepsilon_q^{vp}$ viscoplastic strain increments:

$$\omega_{vp} = -\frac{\Delta \varepsilon_p^{vp}}{\Delta \varepsilon_q^{vp}}$$ (6.94)

If triaxial conditions are considered, the viscoplastic dilatancy $\omega_{vp}$ can be evaluated directly from the ratio between the radial viscoplastic increment $\Delta \varepsilon_r^{vp}$ and the axial viscoplastic strain increment $\Delta \varepsilon_a^{vp}$ as:

$$\omega_{vp} = -\frac{3}{2} \cdot \frac{1 + \frac{\Delta \varepsilon_r^{vp}}{\Delta \varepsilon_a^{vp}}}{1 - \frac{\Delta \varepsilon_r^{vp}}{\Delta \varepsilon_a^{vp}}}$$ (6.95)

If, for all the creep tests performed on clay shales, the radial strain is plotted versus the axial strain, a non-homogeneous behaviour is observed as illustrated in Figure 6.30. Even neglecting the decreasing phases of the radial strain, the ratio between the radial and the axial creep strains are shown to vary between -0.32 and -1.05. The corresponding value of the viscoplastic dilatancy varies between -0.41 and 0.80, with the negative values being associated to a decrease of the volume of the sample and the positive ones to its increase. Neglecting the RTC5 test, which differs significantly from the other two tests, a value of the viscoplastic dilatancy $\omega_{vp}$ equal to -0.41 can be assumed (Table 6.2).

At this point it is necessary to evaluate the initial viscoplastic hardening level $\alpha_{vp,0,i}$ for each creep test $i$ performed. This quantity can be estimated from the stress level $q_{ini,i}, p_{ini,i}$, which defines the onset of viscoplastic deformations, by using Equation (6.26) and the definition of the viscoplastic yield surface ($f_{vp,0,i} = 0$):

$$\alpha_{vp,0,i} = \frac{q_{ini,i}}{p_{ini,i} + \frac{k_p}{\alpha_p}}$$ (6.96)

If the sample has been subjected for a long time to a constant state of stress, which has never been exceeded in the past, it is reasonable to assume that the time dependent process is completed, and the stress point is located on the viscoplastic yield surface. Therefore, this state of stress represents the stress threshold for development of viscoplastic deformations $q_{ini}, p_{ini}$. 
From the laboratory tests on clay shales, no experimental evidence of stress thresholds for the development of viscoplastic deformations is available. Also the in-situ state of stress to evaluate these thresholds is not known. Therefore, the initial viscoplastic hardening level $\alpha_{vp,0,i}$ is assumed to be equal to zero for all the creep tests (Table 6.4). The corresponding values of the initial viscoplastic overstress function $f_{vp,o,i}$ are equal to the deviatoric state of stress (Table 6.4).

To identify the remaining four viscoplastic parameters, only the axial creep strains from triaxial laboratory creep tests are taken into account, as the axial measurements are generally more reliable and accurate than the radial ones. If the logarithm of the axial creep strain rate is plotted versus the logarithm of time, the diagram of Figure 6.31 is obtained. Then, a fitting procedure can be applied to the characteristics equations of the SHELVIP model illustrated in Section 6.5.2, with reference to the existence of the “short-term asymptote” and the “long-term asymptote”.

By using Equation (6.35), the parameter $m_i$ can be evaluated for each creep test as the inverse of the slope of the linear interpolation of the experimental data for high values of time $t$. Figure 6.31 shows this procedure for the creep tests on clay shales and Table 6.4 gives the $m_i$ values obtained. The constitutive
6.8. Analytical calibration procedure

parameter $m$ is then chosen as the arithmetic mean of all the parameters $m_i$.

**Table 6.4:** Parameters used in the calibration of the creep tests on the clay shales (time in days and pressure in kPa)

<table>
<thead>
<tr>
<th>Test</th>
<th>RTC3</th>
<th>RTC4</th>
<th>RTC5</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_{\text{creep}}$</td>
<td>290</td>
<td>178</td>
<td>271</td>
<td>-</td>
</tr>
<tr>
<td>$p_{\text{creep}}$</td>
<td>430</td>
<td>422</td>
<td>446</td>
<td>-</td>
</tr>
<tr>
<td>$\alpha_{v_p,0}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>$f_{v_p,0}$</td>
<td>290</td>
<td>178</td>
<td>271</td>
<td>-</td>
</tr>
<tr>
<td>$m$</td>
<td>0.856</td>
<td>1.085</td>
<td>0.841</td>
<td>0.928</td>
</tr>
<tr>
<td>$\chi$</td>
<td>5.960E-4</td>
<td>3.020E-4</td>
<td>5.540E-4</td>
<td>-</td>
</tr>
<tr>
<td>$n$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1.411</td>
</tr>
<tr>
<td>$\theta$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.010E-7</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.222E-5</td>
<td>2.516E-6</td>
<td>1.110E-5</td>
<td>8.615E-6</td>
</tr>
<tr>
<td>$l$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>30.31</td>
</tr>
<tr>
<td>$\gamma_{\text{corrected}}$</td>
<td>1.138E-5</td>
<td>1.086E-5</td>
<td>1.068E-5</td>
<td>1.097E-5</td>
</tr>
</tbody>
</table>

**Figure 6.31:** Axial creep strain rate versus time in a logarithmic diagram for the creep tests on clay shales
If both terms of Equation (6.34) are divided by the quantity $t^{-\frac{1}{m}}$, one obtains:

\[
\chi = \frac{\varepsilon_{vp}^a |_{\infty}}{t^{-\frac{1}{m}}} = \gamma \cdot l^{-\frac{1}{m}} \cdot q^n \cdot \left( 1 - \frac{\omega_{vp}}{3} \right)
\]  

(6.97)

Therefore, as shown in Figure 6.32, for each test the axial strain rate divided by $t^{-\frac{1}{m}}$ is plotted versus time into a logarithmic diagram and the quantities $\chi_i$ are evaluated as the arithmetic mean of the experimental data, by considering the same fitting range given in Figure 6.31. The values of $\chi_i$ are shown in Table 6.4.

Equation (6.97) can be rewritten into a logarithmic form, to give:

\[
\log(\chi) = n \cdot \log(q) + \log(\theta) \quad \text{with} \quad \theta = \gamma \cdot l^{-\frac{1}{m}} \cdot \left( 1 - \frac{\omega_{vp}}{3} \right)
\]  

(6.98)

As illustrated in Figure 6.33, the logarithm of the quantities $\chi_i$ is plotted versus the logarithm of the deviatoric stresses $q_i$. Then $n$ and $\theta$ can be calculated respectively as the slope and the exponential with base 10 of the intercept with the y-axis of the linear interpolation of the experimental data. The results obtained for $n$ and $\theta$ are reported in Table 6.4.
6.8. Analytical calibration procedure

Figure 6.33: Procedure for the determination of the parameters \( n \) and \( \theta \) for the creep tests on clay shales

The constitutive parameter \( \gamma \) can be evaluated from the “short-time asymptote” of Equation (6.33) as:

\[
\gamma = \frac{\dot{\varepsilon}_{vp}^n|0}{f_{vp,0}^n \cdot \left(1 - \frac{\omega_{vp}}{3}\right)}
\]  

(6.99)

As shown in Figure 6.34, for each creep test the axial strain rate divided by \( f_{vp,0}^n \cdot \left(1 - \frac{\omega_{vp}}{3}\right) \) is plotted versus time into a logarithm diagram. The parameter \( \gamma_i \) is the evaluated for each test as the arithmetic mean of the experimental data for very small values of time \( t \). The values of \( \gamma_i \) are shown in Table 6.4, where also given is the value of \( \gamma \) as the arithmetic mean of all the \( \gamma_i \) values.

Finally the remaining parameter \( l \) can be calculated from Equation (6.98), based on the values of \( \gamma \) and \( \theta \):

\[
l = \left[\frac{\gamma}{\theta} \cdot \left(1 - \frac{\omega_{vp}}{3}\right)\right]^m
\]  

(6.100)

With the values derived for the constitutive parameters given in Table 6.2, Equations (6.28) and (6.29) can be used to compare the computed creep curves with those obtained from triaxial test on clay shales as shown in Figure 6.35. This comparison results to be unsatisfactory. This is due to the fact that the calibration has been performed by using the rates of creep strains instead of
Figure 6.34: Procedure for the determination of the parameter $\gamma$ for the creep tests on clay shales as a first approximation

Figure 6.35: Comparison of laboratory and modelled creep curves for clay shales, by using the first approximation set of parameters
the creep strains. In order to reduce the discrepancy, the following procedure has been adopted.

The fluidity parameter \( \gamma \) is obtained from Equations (6.28) and (6.29) as follows:

\[
\gamma = \frac{\varepsilon_{vp}^{a}}{\tau(t)}
\]  

(6.101)

where:

\[
\tau(t) = \begin{cases} 
\frac{1}{l} \cdot m \cdot q^n \left\{ lt + \left( \frac{q}{f_{vp,0}} \right)^{m \cdot n} \right\}^{\frac{m-1}{m}} + \\
- \left( \frac{q}{f_{vp,0}} \right)^{n(m-1)} \left( 1 - \frac{\omega_{vp}}{3} \right) & \text{for } m \neq 1 \\
\frac{1}{l} \cdot q^n \cdot \ln \left[ 1 + t \cdot \left( \frac{q}{f_{vp,0}} \right)^{-n} \right] \cdot \left( 1 - \frac{\omega_{vp}}{3} \right) & \text{for } m = 1 
\end{cases}
\]  

(6.102)

Then, as shown in Figure 6.36, for each creep test the axial strain \( \varepsilon_{vp}^{a} \) divided by \( \tau(t) \) is plotted versus time. The parameter \( \gamma_i \) can be obtained for each curve as the corresponding arithmetic mean value derived, as given in Table 6.4. Finally \( \gamma \) is chosen as the arithmetic mean of all the parameters \( \gamma_i \). Based on the new value of the fluidity parameter \( \gamma \), the computed and measured creep curves compare extremely well, as illustrated in Figure 6.37. Also very satisfactory is the comparison of computed and measured values of creep strain rate versus time as shown in Figure 6.38.

If a numerical optimization fitting procedure is applied to the creep tests available, by using this set of viscoplastic parameters as first-trial values, a new set of parameters is obtained (Table 6.3). The values of \( m \) and \( n \), which define respectively the shape of the creep curves and the load dependency, are very similar to those already available. The values of the parameters \( \gamma \) and \( l \) differ more significantly from the previous ones. It is noted that \( m \) and \( n \) are power of the model, and, therefore, a little change of their values can produce a very important change of both \( \gamma \) and \( l \), which are multiplier coefficients. If the creep curves based on the new parameters are plotted in Figure 6.39, an even better approximation to the laboratory data is obtained.
Chapter 6. The SHELVIP constitutive model

Figure 6.36: Procedure for the calibration of the parameter $\gamma$ for the creep tests on clay shales, by using the creep strains

Figure 6.37: Comparison of laboratory and modelled creep curves for clay shales, by using the final set of parameters
6.8. Analytical calibration procedure

**Figure 6.38:** Comparison of laboratory and modelled strain rate curves for clay shales, by using the final set of parameters

**Figure 6.39:** Comparison of laboratory and modelled creep curves for clay shales, by using the set of parameters determined by numerical fitting
Chapter 6. The SHELVIP constitutive model

6.9 Calibration of tests on coal samples

This section describes the calibration of the SHELVIP constitutive model on the experimental results of the laboratory tests on coal, as described in Chapter 5.

First, the viscoplastic hardening level of the material before testing has been evaluated with reference to all the tests performed. It is important to remember that the viscoplastic hardening level $\alpha_{vp}$ is a “state parameter” of the material and cannot be considered as a “constitutive parameter” of the model.

Then, the calibration of the constitutive parameters has been carried out with reference to: (1) elastic parameters, (2) plastic parameters, (3) viscoplastic parameters. While the calibration of the elastic and plastic parameters has been done with reference to all the tests, the calibration of the viscoplastic parameters has been applied to single tests.

It is important to note that most of the observed features of the time dependent behaviour of coal can be reproduced correctly by the SHELVIP model. This is not true for the tertiary phase of creep, which however was not intended to be reproduced by the model in the present implementation.

Generally, the calibration of the parameters of the SHELVIP model can be done in two different ways, as described in Section 6.8: (1) by means of an analytical or semi-analytical fitting, (2) by means of a numerical optimization procedure. For simplicity reasons, in this section only the numerical fitting is presented.

6.9.1 Initial viscoplastic hardening level

Because of the few creep tests performed on coal at low deviatoric stress levels, it has not been possible to estimate the viscoplastic hardening level $\alpha_{vp}$ of the material before testing. The lack of direct measurements of the in situ state of stress does not help the evaluation of this state variable.

Therefore the initial viscoplastic hardening level has been conventionally assumed equal to zero. This is equivalent to say that the material has not been subjected in the past to a deviatoric state of stress, that could cause the development of viscoplastic deformations.

6.9.2 Elastic parameters

As pointed out in the previous sections, rigorously the elastic modulus $E$ should be evaluated from unloading phases of triaxial tests. Because of the limited number of available specimens, it has not been possible to perform unloading triaxial tests. Therefore, the elastic modulus $E$ has been evaluated as an average of the loading modulus, as shown in Figure 5.55. The assumed value is given in Table 6.5.
Table 6.5: Elastic parameters of the SHELVIP model for coal

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus</td>
<td>$5000$ MPa</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$0.3$</td>
</tr>
</tbody>
</table>

Because of the lack of accurate radial displacement measurements, the Poisson’s ratio has been conventionally assumed equal to 0.3.

### 6.9.3 Plastic parameters

The constitutive parameters that define the shape of the Drucker-Prager plastic yield surface have been evaluated from the Mohr-Coulomb strength parameters, presented in Section 5.9.1, with reference to the circumscribing case (Equation (6.10)). They are given in Table 6.6. Because of the lack of direct tensile strength tests the tensile strength $\sigma_t$ of the material has been conventionally assumed equal to 0.1 MPa.

Table 6.6: Plastic parameters of the SHELVIP model for coal

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friction angle $\phi$</td>
<td>$34.29$ $^\circ$</td>
</tr>
<tr>
<td>Cohesion $c$</td>
<td>$3.52$ MPa</td>
</tr>
<tr>
<td>Tensile strength $\sigma_t$</td>
<td>$0.1$ MPa</td>
</tr>
<tr>
<td>Strength slope $\alpha_p$</td>
<td>$1.39$</td>
</tr>
<tr>
<td>Strength intercept $k_p$</td>
<td>$7.16$ MPa</td>
</tr>
<tr>
<td>Tensile strength $\sigma_t$</td>
<td>$0.1$ MPa</td>
</tr>
<tr>
<td>Plastic dilatancy $\omega_p$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Since it has not been possible to evaluate the ratio between the volumetric and deviatoric plastic strain increments, the plastic dilatancy has been assumed conventionally to be equal to zero.

### 6.9.4 Viscoplastic parameters

This section describes the calibration of the viscoplastic parameters of the SHELVIP model, which has been performed with reference to single tests. Only the calibration of the viscoplastic dilatancy parameter $\omega_{vp}$ has been made with reference to all the tests. This is due to the fact that it has been possible to evaluate with accuracy the radial displacement of the specimen only for few time dependent tests.
Viscoplastic dilatancy

With reference to Equation (6.95), the radial strains have been plotted versus the axial strain for all the creep tests on coal (only if the measurement of the radial deformations is correct). The obtained diagram is depicted in Figure 6.40.

![Figure 6.40: Radial creep strains versus axial creep strains for the triaxial creep tests on coal](image)

All these curves have been then interpolated with a straight line passing through the origin of the coordinate system. This line is characterized by a slope of $-0.987$. The viscoplastic dilatancy has been then evaluated from this value by using Equation (6.95). A value of 0.735 has been obtained.

It is noteworthy to observe that the time dependent deformations of coal do not develop without volume change, but they are characterized by a significant viscoplastic dilatancy. The volume of the sample increases during the creep tests.

Creep tests A17b

In order to calibrate the remaining four viscoplastic parameters of the SHELVIP model, the two creep phases (point C and D of Figure 5.25) of triaxial test A17b
6.9. Calibration of tests on coal samples

have been considered. The calibration has been done by using the analytical closed-form solution for creep given by Equations (6.28) and (6.29).

According to Section 6.9.1 and by observing that the loading rate before the first creep phase is rather large, the initial viscoplastic hardening level $\alpha_{vp,0}$ at the beginning of the first creep phase has been assumed equal to zero.

As can be seen in Figure 5.30, the time dependent phenomenon can be considered practically completed at the end of the first creep curve. This means that the point, which represents the current state of stress at the end of creep, is located exactly on the viscoplastic yield surface. Therefore, the initial viscoplastic hardening level $\alpha_{vp,0}$ of the second creep curve can be easily evaluated by the stress state of the first creep phase by means of Equation (6.96).

The fitting of the analytical expression on the experimental data has been

<table>
<thead>
<tr>
<th>Table 6.7: Viscoplastic parameters of the SHELVIP model calibrated on the creep phases of test A17b (time in days and pressure in kPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluidity parameter $\gamma$</td>
</tr>
<tr>
<td>Shape factor $m$</td>
</tr>
<tr>
<td>Load dependency $n$</td>
</tr>
<tr>
<td>Time stretching $l$</td>
</tr>
<tr>
<td>Viscoplastic dilatancy $\omega_{vp}$</td>
</tr>
</tbody>
</table>

Figure 6.41: Comparison between the experimental data and the fitting curves of SHELVIP model for the creep phases of test A17b
performed by means of a numerical optimization procedure (least square algorithm) written in the MatLab code. Table 6.7 gives the viscoplastic parameters obtained for the SHELVIP model.

Figure 6.41 illustrates a comparison between the experimental curves and the fitting curves of the SHELVIP model. As shown, the final fitting is indeed excellent.

**Creep tests A17a**

The calibration of the viscoplastic parameters of the SHELVIP model has been made with reference to the first four creep phases of test A17a (points G, H, I, J of Figure 5.33). The last creep phase (point K), during which the tertiary phase of creep has developed, has not been considered because the SHELVIP model is not able to reproduce the creep rupture. Also in this case the analytical solution for creep given by Equations (6.28) and (6.29) has been used.

The main assumption introduced is that the time dependent phenomenon can be considered practically completed at the end of the previous stress relaxation phase. Therefore, the initial viscoplastic hardening level $\alpha_{vp,0}$ at the beginning of the first creep phase has been calculated with reference to Equation (6.96), by considering the stress level at the end of the relaxation phase.

Contrary to the previous calibration procedure, in this case the creep phenomenon cannot be considered completed at the end of the first creep phase. Therefore, the initial viscoplastic hardening level cannot be determined from the state of stress of the previous creep phases. It is necessary to determine the analytical expression of the viscoplastic hardening for creep, by deriving the differential expression of viscoplastic hardening given by Equation (6.23). It is possible to obtain:

$$\alpha_{vp} = \frac{q - \left\{ l \cdot t + \left[ \frac{q}{q - \alpha_{vp,0} \cdot \left( p + \frac{k_p}{\alpha_p} \right)} \right]^{m-n} \right\}^{1/n}}{p + \frac{k_p}{\alpha_p}}$$

The fitting of the analytical expression on the experimental data has been performed by means of a numerical optimization procedure (least square algorithm) written in the MatLab code. Table 6.8 gives the obtained viscoplastic parameters of the SHELVIP model.

Figure 6.42 illustrates a comparison between the experimental data and the SHELVIP model. As shown, the fitting is excellent.
6.9. Calibration of tests on coal samples

Table 6.8: Viscoplastic parameters of the SHELVIP model calibrated on the creep phases of test A17a (time in days and pressure in kPa)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluidity parameter $\gamma$</td>
<td>$5.11 \times 10^{-25}$</td>
</tr>
<tr>
<td>Shape factor $m$</td>
<td>1.051</td>
</tr>
<tr>
<td>Load dependency $n$</td>
<td>6.779</td>
</tr>
<tr>
<td>Time stretching $l$</td>
<td>$7.65 \times 10^{10}$</td>
</tr>
<tr>
<td>Viscoplastic dilatancy $\omega_{vp}$</td>
<td>0.735</td>
</tr>
</tbody>
</table>

Figure 6.42: Comparison between the experimental data and the fitting curves of the SHELVIP model for the creep phases of test A17a

Relaxation test A17a

The calibration of the viscoplastic parameters of the SHELVIP model has been made with reference to the relaxation phase of test A17a (stress path E→F of Figure 5.33). Because it is not possible to find an analytical solution of the SHELVIP model for stress relaxation, the incremental solution given by Equations (6.39) and (6.40) has been adopted.

The initial viscoplastic hardening level $\alpha_{vp,0}$ at the beginning of the stress relaxation phase has been assumed equal to zero.

The fitting of the numerical solution on the experimental data has been performed by means of a numerical optimization procedure (least square algorithm) written into the MatLab code. Table 6.9 shows the viscoplastic parameters of...
Chapter 6. The SHELVIP constitutive model

Table 6.9: Viscoplastic parameters of the SHELVIP model calibrated on the stress relaxation phase of test A17a (time in days and pressure in kPa)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluidity parameter (γ)</td>
<td>1.31E-43</td>
</tr>
<tr>
<td>Shape factor (m)</td>
<td>1.142</td>
</tr>
<tr>
<td>Load dependency (n)</td>
<td>10.236</td>
</tr>
<tr>
<td>Time stretching (l)</td>
<td>4.82E+07</td>
</tr>
<tr>
<td>Viscoplastic dilatancy (ωvp)</td>
<td>0.735</td>
</tr>
</tbody>
</table>

Figure 6.43: Comparison between the experimental data and the fitting curves of SHELVIP model for the relaxation phase of test A17a

Figure 6.43 compares very satisfactorily the experimental data and the fitting curve of the SHELVIP model.

Creep and relaxation

The SHELVIP model, like all the time dependent constitutive models presented in literature, is based on the correspondence principle, that states that the creep and stress relaxation phenomena are governed by the same fundamental equations. This is generally not true: it is quite difficult that a set of constitutive parameters, calibrated on creep tests, can also describe a stress relaxation test, and vice versa.
6.9. Calibration of tests on coal samples

However, the real behaviour of the rock mass during tunnel excavation is something intermediate between pure creep and pure relaxation. Therefore it is very important that the chosen constitutive parameters can describe well both the creep and the stress relaxation phenomena.

For this reason it has been tried to calibrate the constitutive parameters of the SHELVIP model on the results of a creep and a stress relaxation test. The considered creep test is the first creep phase of test A17d (point A of Figure 5.46) which has been carried out without confining pressure. The boundary conditions of this test can be considered very similar to the in situ conditions for a tunnel with very low confinement. The considered stress relaxation test is the relaxation phase of test A17a (stress path E→F of Figure 5.33).

Table 6.10: Viscoplastic parameters of the SHELVIP model calibrated on a creep and relaxation tests (time in days and pressure in kPa)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluidity parameter $\gamma$</td>
<td>4.21E-04</td>
</tr>
<tr>
<td>Shape factor $m$</td>
<td>0.705</td>
</tr>
<tr>
<td>Load dependency $n$</td>
<td>0.181</td>
</tr>
<tr>
<td>Time stretching $l$</td>
<td>5.902</td>
</tr>
<tr>
<td>Viscoplastic dilatancy $\omega_{vp}$</td>
<td>0.735</td>
</tr>
</tbody>
</table>

Figure 6.44: Calibration on creep and relaxation. Comparison between the experimental data and the fitting curves of SHELVIP model for creep
Chapter 6. The SHELVIP constitutive model

Figure 6.45: Calibration on creep and relaxation. Comparison between the experimental data and the fitting curves of SHELVIP model for stress relaxation

The analytical closed form solution of Equations (6.28) and (6.29) has been used to describe the behaviour of SHELVIP model for creep, while the numerical incremental solution given by Equations (6.39) and (6.40) has been adopted for stress relaxation. The fitting of the experimental data has been performed by means of a numerical optimization procedure (least square algorithm) written in the MatLab code. Table 6.10 shows the viscoplastic parameters of the SHELVIP model which are derived.

Figures 6.44 and 6.45 illustrate the comparison between the experimental data and the fitting curve of the SHELVIP model for the creep test and for the relaxation test respectively. It is noted that the fitting results are excellent for the creep test, whereas stress relaxation is represented less satisfactorily. In all cases, it is clear that the SHELVIP model can reproduce well the final relaxed state, which is the most important behavioural feature of stress relaxation.

Considerations

It is quite difficult to compare the different sets of viscoplastic constitutive parameters, because in the SHELVIP model the parameters are connected each other by means of potential expressions: a little change of one parameter can cause a very wide change of the others. This is true especially for the fluidity parameter $\gamma$ and the time stretching $l$. 

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6.10 Conclusions

It is noteworthy to observe that all the calibrations are characterized by a value of $m$ very similar and close to one. Therefore, it is possible to state that all the creep and relaxation tests are characterized by a unique evolution shape.

The parameter $n$ that defines the dependency from the applied load can change significantly. This is probably due to: (1) the heterogeneity of the specimens, (2) the complexity of the performed tests, (3) the impossibility to evaluate correctly the initial viscoplastic hardening level, (4) the load dependency of the sample, that, as shown in Figure 5.57, seems to depend on the distance from the failure surface and not on the applied state of stress $q$.

6.10 Conclusions

The SHELVIP model (Stress Hardening ELastic VIscous Plastic) is a novel time dependent constitutive law that has been proposed with the intent to describe the viscoplastic behaviour of soils/rocks, with particular attention to tunnel excavation and squeezing conditions.

A review of the existing constitutive models has highlighted the lack of a suitable law that can reproduce correctly all the features of time dependence that play an important role during tunnel excavation, in a simple manner. This is the fundamental reason that led to the formulation of a new constitutive model, that is able to take into account the most important aspects of creep and plasticity in a simple way.

The formulation of the SHELVIP model has been based on: (1) the constitutive laws proposed in past, (2) the experimental evidences reported in literature, (3) the theoretical and (4) numerical requirements, and (5) the experimental results of the laboratory test on coal.

The SHELVIP model is a simple extension of the classical theory of elastoplasticity and of the viscoplastic overstress theory of Perzyna. It couples together: (1) a linear elastic law, (2) a viscoplastic flow rule with stress hardening, (3) a classical non-hardening, non-associated plastic flow rule.

According to the classical theory of elastoplasticity, the plastic strains develop only when the stress point reaches the plastic yield surface defined by the Drucker-Prager criterion. The viscoplastic strains develop only if the effective stress state exceeds a viscoplastic yield surface which is also defined by the Drucker-Prager criterion and is internal to the plastic yield surface. The viscoplastic yield surface can harden with time as a function of the applied overstress state (stress hardening). The most important advance of the stress hardening rule is that the viscoplastic hardening level of a material can be estimated only by determining experimentally the stress level which corresponds to the beginning of time dependent deformations.
Chapter 6. The SHELVIP constitutive model

The overall number of constitutive parameter is 11: 2 elastic, 4 plastic and 5 viscoplastic. It is a quite small number if compared to other viscoplastic models. Each aspect of the behaviour of the model is controlled by a single parameter.

An analytical closed form solution of the SHELVIP model can be found for the case of creep. This solution is very useful in order to calibrate the constitutive parameters of the model on laboratory creep tests. For constant strain rate conditions only a incremental numerical solution can be found.

By observing the behaviour of the model with reference to classical time dependent tests, it should be noted that the SHELVIP model can reproduce almost all aspects of creep. The only two aspects that the model cannot reproduce satisfactorily are: (1) the dependency of the strength envelope from the applied strain rate, (2) the tertiary phase of creep. In the future research, these aspects should be introduced into the SHELVIP model by a viscoplastic hardening process.

The SHELVIP model has been implemented into the finite difference code FLAC, in order to allow to perform numerical analyses of geotechnical problems. The implementation has been made by means of a library written with C++ in order to minimize the computational time. It is very important to remember that for complex analyses and back analyses the required time is extremely important and must be reduced to a minimum. The numerical implementation has been verified by means of analytical and incremental solutions.

An analytical calibration procedure of the constitutive parameters of the SHELVIP model on the results of creep tests has been presented with reference to three triaxial creep tests performed on clay shales.

Finally, the SHELVIP model has been calibrated by using the triaxial tests performed on coal specimens as described in Chapter 5. It has been shown that the creep and relaxation tests can be described very satisfactorily.
Chapter 7

Application of SHELVIP model to the Saint Martin La Porte tunnel

7.1 Introduction

This chapter describes the back analysis of a representative cross section of the Saint Martin La Porte access tunnel which underwent a very severe squeezing behaviour during excavation. The main scope is to highlight the potentials of the newly developed SHELVIP model for the numerical analysis of tunnels in severe to very severe squeezing conditions. Therefore, this case study is not taken to give a comprehensive description of all the aspects involved in the design of tunnels in squeezing rock.

7.2 Saint Martin La Porte tunnel

The description of the tunnel and geological conditions of the rock mass has been illustrated in Section 5.2.1. In the present section only the construction sequence and the monitoring data in the cross section of interest (chainage 1280 to 1400 m) are described.

7.2.1 Construction sequence

Several support systems were used in the Carboniferous zone. However, it became soon apparent that a stiff support ("heavy method" or "resistance principle") would not be feasible in the severe squeezing conditions encountered. The design concept chosen was to accommodate the large deformations, which are expected to develop in squeezing conditions, allowing the support to yield
and using systematic face reinforcement (“light method” or “yielding principle”).
The support system implemented first (P7.3) consisted of yielding steel ribs
with sliding joints (TH44/58, Toussaint-Heintzmann type), anchors and a thin
shotcrete layer in a horseshoe profile (Figure 7.1). This is the support system
adopted in the cross section to be analysed in the following.

![Figure 7.1: Transversal and longitudinal section of the Saint Martin La Porte tunnel](image)

The construction sequence is as follows:

- excavation and concurrent placement of an umbrella of 20 self-drilling
  bolts, 8 m long, with an overlapping of 2 m and a spacing of 50 cm;

- reinforcement of the tunnel face by means of fibre-glass dowels, and a
  5-10 cm thick shotcrete layer both on the face and on the perimeter;
  installation of 10 swellex bolts at the crown;

- installation of 34 self-drilling bolts, 8 m long, all around the perimeter;
  installation of the TH steel ribs with sliding joints with spacing of 1 m;
  reinforcement of the footing with a 20 cm thick shotcrete layer

Figure 7.2.a gives a photograph of the tunnel face and of the support system.
These sections of the tunnel underwent very large deformations with con-
vergences up to 2 m and later needed be re-profiled (Figure 7.2.b). Some local
failure and instabilities developed and the support system were damaged in
several points (Figure 7.3).

In order to improve the working conditions and to control deformations a
novel support system (DSM XX) has been implemented with a near circular
cross section (Barla et al., 2007b,c, 2008; Rettighieri et al., 2008).
7.2. Saint Martin La Porte tunnel

Figure 7.2: Tunnel face and support system 7.3 (a), and re-profiling after squeezing deformations (b)

Figure 7.3: Local failure (a) and damage of the support system (b)

7.2.2 Monitoring data

The excavation sequence adopted in the tunnel between chainage 1280 and 1400 m, where the support system installed is type P7.3 (Figure 7.1), is shown in Figure 7.4. It is observed that, due to the difficulties encountered during excavation along this tunnel length, a number of stops of face advance took place. Being the cross section chosen for back analysis at chainage 1311 m, the time interval of interest is also shown in the same Figure 7.4.

Systematic tunnel observation and performance monitoring concurrent with excavation has been carried out by measuring the following:
Chapter 7. Application of SHELVIP model to the Saint Martin La Porte tunnel

- convergences, with optical targets placed along the tunnel perimeter;
- radial displacements, by means of multi-position borehole extensometers;
- longitudinal displacements ahead of the tunnel face, by means of extrusometers.

Convergences

The measurement of convergence has been performed by using 5 optical targets placed along the tunnel perimeter on the TH steel ribs, as depicted in Figure 7.5. The mean distance between a measurement section and the following one is approximately 5-10 m. This system has allowed to promptly evaluate the tunnel convergence during excavation in the Carboniferous Formation. An example of the monitored convergences is illustrated in Figure 7.6 for the section at chainage 1311.

It is possible to make the following observations:

- The magnitude of the convergences is very high, up to 2m.
- The deformational behaviour of the tunnel is highly anisotropic: the greatest convergences are along arrays 1-3 and 1-5. Generally the tunnel
7.2. Saint Martin La Porte tunnel

Figure 7.5: Position of the optical targets for the measurement of convergences

Figure 7.6: Convergences at chainage 1311
section is characterized by an inclined preferential direction of deformation (from NNE to SSO).

- The tunnel continues to deform even if the excavation operations are stopped. This implies necessarily a time dependent behaviour of the rock mass.

- The resumption of the excavation causes an increase of the rate of deformation, even if the face is quite distant from the considered section. The influence zone of the tunnel ahead is very large.

- There is no significant asymptotic behaviour during the period of observation of 160 days. This means that the deformations of the tunnel occur during a very long period of time.

Figure 7.7 shows the convergences along the horizontal array 1-5 with the tunnel face being 15 m and 30 m ahead of the monitoring section. It is possible to notice that the convergences, that develop when the tunnel face advances from 15 m to 30 m to the considered section, and that are probably connected to the time dependent behaviour of the rock mass, are quite important.

![Figure 7.7: Convergences measured along arrays 1-5 at 15 m and 30 m from face](image)
Radial displacements

The section at chainage 1330 m has been instrumented with 6 radial multi-position borehole extensometers, 18 m long, positioned as illustrated in Figure 7.8.

![Radial multi-position borehole extensometers](image)

Figure 7.8: Radial multi-position borehole extensometers

Figure 7.9 shows the radial displacements measured by the fourth extensometer, located at the right sidewall of section 1330, for 5 different dates (distant 29 days from each other).

It is possible to observe that the radial displacements are quite high and the zone of the rock mass, which is undergoing these relevant deformations, is extended in depth more than the length of the extensometer itself.

A very interesting representation of the monitored data of the radial extensometer can be obtained by plotting the radial displacement at the tunnel boundary for a fixed times in a radial diagram (Figure 7.10).

It should be noticed that the deformation of the section is characterized by a relevant anisotropy, with a sub-horizontal preferential direction.
Figure 7.9: Radial displacement of extensometer 4 for the section at chainage 1330

Figure 7.10: Radial displacement at tunnel boundary for section at chainage 1330
Longitudinal displacement

As shown in Figure 7.11, the longitudinal displacements ahead of the tunnel face have been measured by means of a 36 m long extrusometer installed at chainage 1345. The total duration of the measurements has been approximately 3 months.

By analysing this diagram it is possible to identify the effect of the face advance (measurement from 18/01/06 to 14/02/06) and the effect of time (measurements from 14/02/06 to 17/03/06 when the face stopped at the chainage 1368). During the face advance the mean displacement of the face is 5 cm; during the stop of the excavation this displacement increases up to 10 cm. The depth of the zone undergoing this longitudinal displacement is approximately equal to 5-6 m.

It can be noted that, even if the displacement rate is not negligible, the magnitude of the longitudinal displacements measured by the extrusometer is considerably smaller than expected (theoretically, the longitudinal displacement is equal to 2/3 of the radial displacement measured at 2.5 diameters from the face).
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7.3 Numerical back analysis

The main steps leading to the application of the SHELVIP model to the Saint Martin La Porte tunnel by means of the FLAC code (ITASCA, 2006) are described in the following.

The aim of the numerical back analysis is to evaluate the ability of the novel SHELVIP model to describe the squeezing deformations observed during tunnel excavation, giving appropriate attention to the stress-path development in the tunnel surround. Therefore, the attention is focused more on the simulation of the time dependent behaviour of the rock mass than on the representation of the real geometrical and structural characteristics of the tunnel.

The cross section of the tunnel taken for purpose of back analysis, is very suitable because the adopted support system permits the squeezing deformations to develop completely without a significant influence of the lining. Due to this the deformations of the tunnel are connected only to the time dependent behaviour of the rock mass.

7.3.1 Mesh and boundary conditions

Since the main purpose is to study the behaviour of the new SHELVIP constitutive model, a rather simple axi-symmetric model of the tunnel has been adopted. This leads to different advantages:

- It is possible to reproduce easily the three-dimensional influence of the tunnel face, which is known to play a very significant role in squeezing conditions.
- It is possible to simulate the real chronological sequence of excavation, that influences the time dependent deformations of the tunnel.
- The overall time dependent behaviour of the tunnel is not widely influenced by geometrical effects, such as stress concentrations.
- It is possible to use a regular mesh with square elements, in order to minimize the numerical errors and the error propagation.
- It is possible to use a large model in order to neglect the boundary effects.
- The solution time can be reduced to a minimum. This permits to perform a large number of analyses in order to match exactly the time dependent behaviour of the rock mass.

The tunnel cross section is assumed to be circular, with an equivalent radius of 5.5 m. Figure 7.12 shows the FDM mesh adopted. The mesh is composed
of perfectly square elements, with size increasing gradually from 0.5 m to 4 m when moving from the near vicinity of the tunnel outwards, in order to minimize error propagation and solution time. The total size of the mesh (184 m - 92 m) is very large in order to minimize the boundary effects that are very significant in the case of large deformations.

![Figure 7.12: Numerical model of the Saint Martin La Porte tunnel: mesh and boundary conditions](image)

The influence of the primary lining and the reinforcement system has been completely neglected. The boundary conditions, which are shown in Figure 7.12, have been chosen by means of preliminary numerical analyses in order to minimize the numerical error and the solution time. The overburden is approximately 310 m and the initial vertical stress is assumed to be 8.4 MPa, with the stress ratio \((K_0)\) equal to 1 (i.e. hydrostatic conditions). Dried conditions have been considered.

### 7.3.2 Excavation sequence and solution procedure

Particular attention has been posed on the chronological sequence of excavation (face advance) which is considered to influence the time dependent deformational response.

The complex excavation sequence has been simulated by means of 245
computational steps, as illustrated in Figure 7.13. Three excavation phases, with rate of 1, 1.5 and 1 day/m, and three stops of face advance for 32, 19, and 57 days have been considered.

The excavation phase has been simulated numerically by means of discrete advancement steps of 0.5 m, in order not to introduce a too high numerical disturbance into the model. For each step the solution has been obtained by performing successively: (1) a plastic solution, and (2) a viscoplastic solution. This allows to obtain a more stable and correct solution, because the viscoplastic computation is not influenced by the unbalanced state of stress due to the removal of the excavated elements.

7.3.3 Calibration

Preliminary numerical analyses have been performed by using the constitutive parameters evaluated at laboratory scale. In particular, the viscoplastic parameters have been assumed to be the same as calibrated simultaneously on the creep phase of test A17d and on the stress relaxation phase of test A17a on coal, as reported in Section 6.9.4.

The computations performed showed that this set of parameters is not able to reproduce correctly the deformations of the tunnel. This is clearly understandable if one considers the scale effect, the heterogeneity of the rock
7.4 Numerical results

mass, which is composed of several rock types and not only of coal, and the presence of joints.

Therefore, a constitutive parameters calibration has been carried out with reference to the convergences of section at chainage 1311 m (Figure 7.6). A total number of 61 analyses have been performed before an acceptable representation of the monitored response could be obtained, with the constitutive parameters of Table 7.1. It is noted by comparing these values with those given in Table 6.10 that the viscoplastic parameters $n$ and $\omega_{vp}$ have been kept constant.

<table>
<thead>
<tr>
<th>Table 7.1: Constitutive parameters of the SHELVIP model for the Saint Martin La Porte tunnel (time in days and pressure in kPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus $E$</td>
</tr>
<tr>
<td>Poisson’s coefficient $\nu$</td>
</tr>
<tr>
<td>Friction angle $\phi$</td>
</tr>
<tr>
<td>Cohesion $c$</td>
</tr>
<tr>
<td>Tensile strength $\sigma_t$</td>
</tr>
<tr>
<td>Strength slope $\alpha_p$</td>
</tr>
<tr>
<td>Strength intercept $k_p$</td>
</tr>
<tr>
<td>Tensile strength $\sigma_t$</td>
</tr>
<tr>
<td>Plastic dilatancy $\omega_p$</td>
</tr>
<tr>
<td>Fluidity parameter $\gamma$</td>
</tr>
<tr>
<td>Shape factor $m$</td>
</tr>
<tr>
<td>Load dependency $n$</td>
</tr>
<tr>
<td>Time stretching $l$</td>
</tr>
<tr>
<td>Viscoplastic dilatancy $\omega_{vp}$</td>
</tr>
<tr>
<td>Initial viscoplastic hardening $\alpha_{vp}$</td>
</tr>
</tbody>
</table>

It is important to note that the value of parameter $m$ is greater than 1. According to what observed in Section 6.5.1, during the time interval under study (approximately 160 days) the squeezing phenomenon cannot be considered to be completed.

7.4 Numerical results

In this section the results of the numerical analysis performed by using the SHELVIP model and the set of constitutive parameters reported in Table 7.1 on the Saint Martin La Porte tunnel are presented and described.
7.4.1 Convergences

Figure 7.14 compares the results of the numerical analyses and the monitoring data, in terms of radial displacements, for section at chainage 1311 m. Each radial displacement has been calculated from the measured data, by dividing the convergence for the initial length and by multiplying by the equivalent radius (equi-area) of the tunnel. This procedure is necessary to compare the monitored data with the results of the analysis.

![Figure 7.14: Radial displacement at section 1311: comparison of monitored data and computed results](image)

It possible to notice that the curve of the SHELVIP model reproduces very well the mean radial displacement (arrays 1-3, 1-5 and 2-4) over the entire time interval of 160 days, notwithstanding the scattering of the monitoring data due to the heterogeneity and anisotropy of the rock mass. It matches very well the behaviour of the tunnel both during the excavation phases and during the stops of face advance.

Figure 7.14 illustrates the curves obtained by the elastic and elastoplastic analyses using the constitutive parameters reported in Table 7.1. As easily seen, the plastic curve cannot reproduce the behaviour of the tunnel, which is characterized by an important time dependent response.

Figures 7.15 to 7.18 give the comparison of the monitored data and computed results in terms of radial displacement for sections at chainage 1322 m, 1331 m, 1342 m, and 1356 m respectively. It is worth to notice that the SHELVIP model
7.4. Numerical results

**Figure 7.15:** Radial displacements at section 1322: comparison of monitored data and computed results

**Figure 7.16:** Radial displacements at section 1331: comparison of monitored data and computed results
Figure 7.17: Radial displacements at section 1342: comparison of monitored data and computed results

Figure 7.18: Radial displacements at section 1356: comparison of monitored data and computed results
can reproduce in an accurate way the mean radial displacement of the sections at chainage 1322 m, 1331 m and 1342 m. For the section at chainage 1356 m the SHELVIP model overestimates the radial displacement measured during excavation. It is observed that the geological conditions in this section are different from those encountered in the previous sections.

### 7.4.2 Extensometer measurements

Figures 7.19 to 7.22 compare the monitored data and the results of the numerical analysis, in terms of the radial displacements of the extensometer installed at chainage 1330 m for 4 different dates. The time interval between the installation of the extensometer and the first measurement, and between each measurement is constant and equal to 29 days. The numerical analysis performed with the SHELVIP model can reproduce very well the mean curves, notwithstanding the scattering of the monitoring data due to the heterogeneity and anisotropy of the rock mass.

### 7.4.3 Extrusometer measurements

Figure 7.23 illustrates the comparison between the longitudinal displacement measured by means of the extrusometer installed at chainage 1345 (11/12/2005) and the results of the numerical analysis for 10 different dates. The numerical analysis performed with the SHELVIP model reproduces very well the monitored longitudinal displacements of the tunnel for the first two dates (18/01/2006 and 19/01/2006), while it overestimates the remaining measurements both during excavation and the stops of face advance. This is probably due to a change of the properties of the rock mass along the tunnel. It is in fact important to remember that the constitutive parameters of the SHELVIP model have been calibrated on the convergences of section 1311.
Figure 7.19: Radial displacements of the extensometer at section 1330 on 09/01/2006 (installation 11/12/2005): comparison of monitored data and computed results

Figure 7.20: Radial displacements of the extensometer at section 1330 on 07/02/2006 (installation 11/12/2005): comparison of monitored data and computed results
7.4. Numerical results

Figure 7.21: Radial displacements of the extensometer at section 1330 on 08/03/2006 (installation 11/12/2005): comparison of monitored data and computed results

Figure 7.22: Radial displacements of the extensometer at section 1330 on 06/04/2006 (installation 11/12/2005): comparison of monitored data and computed results
Figure 7.23: Longitudinal displacements of the extrusometer at section 1345 (installation 15/01/2006): comparison of monitored data and computed results
7.4.4 Stress-paths and stress variation

Stress-paths

In order to point out some important features of the tunnel behaviour in terms of the stress conditions in the tunnel surround as excavation takes place, the computed stress-paths for typical points A (1 m away from the tunnel contour, Figure 7.12) and B (located at the face, along the tunnel axis, Figure 7.12) for the section at chainage 1331 are shown in Figures 7.24 and 7.25 with reference to the $q$-$p$ plane.

![Stress-paths for point A of section at chainage 1331 during face advancement](image)

Figure 7.24: Stress-paths for point A of section at chainage 1331 during face advancement

For both points, a rather linear trend is observed until the point which represents the state of stress reaches the strength envelope. The element of soil represented by point A yields a short time after the arrival of the face, while the element represented by point B yields long time before. After yielding, the deviatoric stress $q$ and the mean stress $p$ decrease by following exactly the strength line. The final state is defined by $q=5.15$ MPa and $p=4.12$ MPa for point A, while, obviously, the final state of point B is coincident with the origin.
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Figure 7.25: Stress-paths for point B of section at chainage 1331 during face advancement

(by observing this two stress-paths, with reference to the distance of the tunnel face, it is possible to state that most of the stress change occurs when the face is ±5 m far for point A and 7 m far for the point B.

It is very interesting to compare the stress-paths obtained with the SHELVIP model with those obtained by a simple elastoplastic analysis. Both for point A and point B, the effect of the viscoplastic component of the SHELVIP model is to incline the stress path to the right. This is essentially due to the viscoplastic dilatancy that has been assumed greater than zero.

Stress variation

As an additional point of interest for getting a better insight into the stress conditions around the tunnel, the variation of stresses during face advancement is considered in the following.

Figure 7.26 illustrates the variation of the effective stresses \( q \) and \( p \) at the point A versus the position of the tunnel face. The approach of the tunnel...
7.4. Numerical results

Figure 7.26: Variation of the stresses $q$ and $p$ for point A versus the distance of the tunnel face

Figure 7.27: Variation of the stresses $q$ and $p$ for point B versus the distance of the tunnel face
face causes an increase of both the deviatoric stress $q$ and the means stress $p$, that reach the maximum values immediately after the arrival of the face. The departing of the face causes a decrease of both $q$ and $p$, that after some time tends to a constant value. Most of the stress variations occurs between -10 and +10 m.

It is very interesting to observe the effect of the stop of excavation. Two standstill phases are present: one of 32 days with the face 9 m before the considered section and one of 19 days with the face 5 m after. The effect of the first phase is to decrease both the deviatoric stress $q$ and the mean stress $p$, while the effect of the second phase is to increase both $q$ and $p$.

Figure 7.27 shows the variation of $q$ and $p$ versus the distance of the tunnel face for point B. The approach of the tunnel face first causes an increase of both $q$ and $p$, that reach a maximum a few meters before the arrival of the face, and then a sudden decrease while the face arrives at the section. The final values of $q$ and $p$ are obviously equal to zero.

**Stress distribution around the tunnel**

An additional factor of interest for the assessment of tunnel stability in difficult ground conditions during excavation is the behaviour of the rock mass surrounding the tunnel.

Figures 7.28 and 7.29 depict the variation of the deviatoric stress $q$ and the mean stress $p$ versus the radial distance from the tunnel contour for the section at chainage 1330 m for four cases: (1) tunnel face 10 m before the section, (2) arrival of the face, (3) face 10 m after, (4) 118 days after the excavation of the section.

As easily seen, the effect of the face advance is to expand the plastic zone (defined by the peak of the curve) and to decrease the stress acting on the contour of the tunnel. The maximum extension of the plastic zone is 4.72 m, which is approximately equal to one tunnel radius.

Time does not considerably modify the extension of the plastic zone, but increases the maximum values of both the deviatoric stress $q$ and the mean stress $p$, as observed in the previous section. It is very interesting to observe that the effect of time on the radial distribution of the deviatoric stress $q$ is to redistribute the state of stress, by concentrating even more the stress around the limit of the plastic zone.

An extension of the plastic zone, that some authors consider an effect of time dependence, can be taken into account only by modifying the formulation of the SHELVIP model and by introducing a viscoplastic hardening of the plastic yield surface.
7.4. Numerical results

Figure 7.28: Radial distribution of the deviatoric stress $q$

Figure 7.29: Radial distribution of the mean stress $p$
Chapter 7. Application of SHELVIP model to the Saint Martin La Porte tunnel

Distribution along the tunnel

Figures 7.30 and 7.31 illustrate the distribution of the deviatoric stress \( q \) and the mean stress \( p \) along the tunnel axis, immediately after the excavation of the section at chainage 1322 m, and after 32 days of stopping of the excavation.

It is possible to notice that the viscoplastic deformations, that occur during the stopping of the excavation, do not modify the extension of the plastic zone, as observed for the stress distribution across the tunnel. The maximum extension of the plastic zone is 3.70 m, approximately equal to one tunnel radius.

7.5 Conclusions

The Saint Martin La Porte access tunnel, which has been excavated in the Productive Carboniferous Formation by using a very deformable support system, underwent very severe squeezing conditions with convergences up to 2 m.

The monitoring data (convergences, radial and longitudinal displacement) highlight a considerable time dependent behaviour of the rock mass, associated with an important anisotropic response. This type of behaviour has been also observed during laboratory testing on samples of coal taken from the tunnel, as described in Chapter 5.

A series of numerical back analysis have been performed by using the new constitutive model SHELVIP, that has been especially proposed to describe the squeezing behaviour during tunnel excavation in poor rock mass conditions (Chapter 6). The main aim of the analysis is to evaluate the ability of the SHELVIP model to describe the squeezing conditions with reference to a real case study.

A quite simple numerical model has been used to represent the tunnel geometry. This allows some important benefits. The most important is to separate the behaviour of the rock mass from the effects of the geometry and support system.

Preliminary numerical analyses, performed by using the viscoplastic parameters calibrated on the laboratory tests on coal, shows that this set of constitutive parameters is not able to reproduce correctly the time dependent deformations of the tunnel. Therefore a series of calibration analysis has been carried out in order to find an appropriate set of constitutive parameters that can reproduce the monitored displacements. The calibration procedure has been performed with reference to the convergences monitored at chainage 1311.

The results of the numerical analysis have been presented. The SHELVIP model can describe very well all the monitored displacements (convergences, radial and longitudinal displacements) of the tunnel, notwithstanding the scat-
7.5. Conclusions

Figure 7.30: Longitudinal distribution of the stress $q$ along the tunnel axis

Figure 7.31: Longitudinal distribution of the stress $p$ along the tunnel axis
tering of the monitoring data due to the heterogeneity and anisotropy of the rock mass. The tendency is to lightly overestimate the deformation of the final part of the tunnel (from section 1356 m). This is probably due to a small variation of the characteristics of the rock mass between the section of calibration (chainage 1311 m) and the neighbouring sections.

However, it is possible to state that the agreement of the numerical results with the mean monitored data is excellent. Therefore, it is possible to conclude that the ability of the SHELVIP model to describe the squeezing conditions is verified successfully.

In the last part of the chapter the stress distribution around the tunnel contour is described. The results are satisfactory and reliable.

In particular, it should be noticed that the SHELVIP model cannot reproduce the expansion of the yield zone due to viscoplastic damage. This aspect could be taken into account with further modifications of the SHELVIP model by introducing a viscoplastic negative hardening of the plastic yield surface.

Further analyses should be performed in order to evaluate the ability of the SHELVIP model to describe with accuracy the interaction between the rock mass and the support system.
Chapter 8

Summary and conclusions

8.1 Summary

The following main tasks have been accomplished in the present thesis.

- A detailed bibliographic study of time dependent behaviour of geomaterials was performed. First, a complete description of the experimental evidences reported in literature from various authors was given. The main features, which are commonly accepted by the geotechnical community, and a comprehensive approach of time dependence were illustrated with reference to classical triaxial and oedometer laboratory tests. Then, the interest was directed to the description of constitutive models proposed in the last sixty years to describe the viscoplastic behaviour. A comprehensive review was presented in conjunction with a general classification of the models.

- The HPTA (High Pressure Triaxial Apparatus) of the DIPLAB Labora-
tory of the Department of Structural and Geotechnical Engineering of Politecnico di Torino (an experimental apparatus for the study of weak rocks under high confining conditions) was modified and calibrated. A detailed description of this equipment and a brief presentation of the HPBPSA (High Pressure Back Pressure Shear Apparatus) used for the characterization of the material of the Saint Martin La Porte access tunnel is given, with attention to elements of novelties.

- Coal samples, taken as representative of the squeezing behaviour of the Saint Martin La Porte tunnel, which underwent very severe squeezing deformations during excavation, were sampled and taken to the laboratory. Triaxial, oedometer, and direct-shear tests performed in closely controlled
conditions, by using the two equipments described above, allowed to evaluate the most important aspects of the mechanical behaviour of coal, which influences the tunnel response. Particular attention was posed on the time dependent characteristics of the material by means of creep and stress relaxation tests.

- The new SHELVIP (Stress Hardening ELastic Viscous Plastic) constitutive model was developed with the intent to describe the most important aspects of creep that play an important role during tunnel excavation. The mathematical formulation of the model, the behaviour under classical stress paths, the implementation in the finite difference code FLAC, and its validation were described in detail. Finally the calibration of the model on the laboratory tests on coal specimens was presented.

- Numerical back analyses were performed on a representative section of the Saint Martin La Porte tunnel by using the SHELVIP model, with the main scope to highlight the potentials of the model for numerical analysis of tunnels in severe to very severe squeezing conditions.

8.2 Conclusions

It is the purpose of the present chapter to draw some conclusions on the work performed so far. The following aspects will be considered:

- Experimental time dependent behaviour of soils
- Constitutive models for time dependent behaviour of soils
- Testing equipment
- Laboratory testing on coal
- The SHELVIP constitutive model
- Application of SHELVIP model to the Saint Martin La Porte tunnel

8.2.1 Experimental time dependent behaviour of soils

Squeezing may occur in any soil or rock as long as the particular combination of induced stresses and material properties brings some zones of the rock mass beyond the limiting shear stresses, inducing time dependent deformations.

Time dependence results at the laboratory scale in the following phenomena: creep, stress relaxation, strain rate dependence, and apparent preconsolidation. The most interesting aspect is creep, which can be distinguished in primary,
8.2. Conclusions

secondary, and tertiary phases. The tertiary phase is very important in tunnel
design because it leads to a delayed failure of the material.

The bibliographic study of experimental evidences reported in literature
pointed out that it is possible to define a general and well known time depen-
dent pattern which characterizes all the types of soil. Only the stress levels
which define the transition between the different phases of creep are not well
understood.

8.2.2 Constitutive models for time dependent behaviour
of soils

A great number of constitutive models were proposed from different authors
in the last sixty years, with the intent to capture the various time dependent
phenomena. Different approaches were used: empirical models, rheological
models, and general theories. General theories represent the most advanced
aspect of constitutive modelling and describe not only the viscous effects but
also the inviscid (time independent) behaviour of soils. They can include the
most recent aspects of the research on plasticity. The overstress theory of
Perzyna and the Nonstationary Flow Surface (NSFS) belong to this category.

The bibliographic study highlighted the lack of suitable constitutive models
that can describe in a simple way all the most important aspects of creep that
play an important role during tunnel excavation. It is important to notice that
a model which is too complex or too simple cannot be used with confidence
both for research and for design practice.

8.2.3 Testing equipment

One laboratory equipment used in this thesis is the High Pressure Triaxial
Apparatus (HPTA) which has been especially developed for the DIPLAB
Laboratory of the Structural and Geotechnical Engineering Department of
Politecnico di Torino. The main purpose of its development is the interest for
testing weak rocks at high confining pressure, with the accuracy allowed for by
local strain measurement and a fully automatic controlling system.

Most of the attention was posed, developing the new triaxial apparatus, to
a number of special features which make the equipment innovative and well
advanced in many aspects: the high confining and interstitial pressure, the
high stiffness of the load frame, the presence of a balancing cell, the spherical
seat, the local strain measurement, the data acquisition and the control system.
These features make the apparatus a “unicum” in the geotechnical research
field, with a large number of applications.
8.2.4 Laboratory testing on coal

The laboratory testing program pointed out the main characteristics of coal pertaining to the Carboniferous Formation met during excavation of the Saint Martin La Porte access tunnel, which exhibited a very severe squeezing behaviour. A characteristic feature of the ground as observed at the face during excavation is the highly heterogeneous, disrupted and fractured condition of rock mass. The formation is often affected by faulting which results in a degradation of the rock mass conditions. Coal was chosen as representative of the time dependent behaviour, which is supposed to widely influence the squeezing deformations of the rock mass. Samples of coal from a neighbouring borehole, approximately at the same depth of the considered section of the tunnel, were taken to the laboratory, where cylindrical specimens were cut although with great difficulties.

Geotechnical classification highlights that coal can be considered as an intermediate material between soil and rock (high void index, significant Atterberg’s limits). However, the clay percentage is small and no evidence of expansive minerals, which can lead to the swelling phenomenon during tunnel excavation, was found.

Two oedometer-direct shear and four triaxial tests were carried out by means of the HPBPSA and HPTA apparatus, with the intent to evaluate the mechanical behaviour of coal. Particular attention was posed on peak and residual strength, deformability and time dependent behaviour. The aim was to simulate at laboratory scale the tunnel response in the short and long term conditions. The few number of specimens available led to the necessity to perform a great number of stages (strength, creep, stress relaxation) for each test. This allowed one to obtain a great number of information from the same sample, however making it difficult and quite complex to interpret the results obtained. The small number of tests performed is the most important limitation of this work, especially if the heterogeneity of the samples is taken into account. In some occasions it was possible to define only a qualitative behaviour or trend of the material tested.

The peak strength increases with the applied mean stress and can be represented well using the Mohr-Coulomb criterion. The non-linearity near the origin is not very clear because of the lack of tensile strength data and results of triaxial tests under low mean stress.

The deformability of coal under volumetric state of stress highlighted a significant hardening with the increase of the mean stress, which is a typical soil behaviour. As a consequence, the behaviour under deviatoric state of stress pointed out a loading modulus rather larger than the dynamic modulus determined using ultrasonic tests.

The time dependent behaviour of coal was investigated extensively by means of creep and stress relaxation tests performed at different states of stress. It
is noticed that the deferred behaviour is quite significant compared to the instantaneous deformations. A very important aspect of time dependence is the development of the three phases (primary, secondary and tertiary) of creep depending on the stress level applied. Most of the time dependent characteristics of coal follow the experimental evidences reported in literature. Only the load dependency differs considerably.

These observations formed the background for the formulation of a new viscoplastic constitutive model.

8.2.5 The SHELVIP constitutive model

The novel elastoviscoplastic constitutive model SHELVIP (Stress Hardening ELastic VIscous Plastic) was developed with the intent to describe the squeezing phenomenon, that can occur during tunnel excavation in poor rock mass conditions. It was proposed to supply the lack of a suitable constitutive laws that can reproduce all the time dependent aspects of the behaviour of weak rocks which play an important role in tunnel excavation.

The formulation of the SHELVIP model has been based on: (1) the constitutive laws proposed in past, (2) the experimental evidences reported in literature, (3) the theoretical and (4) numerical requirements, and (5) the experimental results of the laboratory tests on coal. The SHELVIP model has been derived from the Perzyna’s overstress theory, by adding a time independent plastic component. According to the classical theory of elastoplasticity, the time independent plastic strains develop only when the stress point reaches the plastic yield surface defined by the Drucker-Prager criterion. The viscoplastic strain develops only if the effective stress state exceeds a viscoplastic yield surface which is also defined by the Drucker-Prager criterion. The hardening of the viscoplastic yield surface is controlled by the overstress state (stress exceeding the viscoplastic surface) by means of a differential relationship (stress hardening). The most important feature of the stress hardening rule is that the viscoplastic hardening level of a material can be estimated only by determining experimentally the stress level which corresponds to the beginning of time dependent deformations.

An analytical closed form solution of the SHELVIP model can be found for the case of creep. This solution is very useful in order to calibrate the constitutive parameters of the model on laboratory creep tests.

Observing the model behaviour with reference to classical time dependent tests, it should be noted that the SHELVIP model reproduces almost all aspects of creep. The only two aspects that the model cannot reproduce satisfactorily are: (1) the dependency of the strength envelope from the applied strain rate, (2) the tertiary phase of creep. In the future research, these aspects should be introduced into the SHELVIP model by a viscoplastic hardening process.
The SHELVIP model was implemented in the commercial finite difference code FLAC by using a library written with the C++ language, which was numerically optimized and then validated by means of analytical or semi-analytical solutions.

Finally, the SHELVIP model was calibrated by using the triaxial tests performed on coal specimens. It was shown that the creep and relaxation tests can be described in an accurate way.

It is possible to state that the SHELVIP model is a rather simple but powerful constitutive law, which can improve the understanding of squeezing, and can assist in the choice of the appropriate excavation and support systems to be adopted.

8.2.6 Application of SHELVIP model to the Saint Martin La Porte tunnel

A series of numerical analyses were carried out using the newly developed SHELVIP model on a representative section of the Saint Martin La Porte access tunnel, with the intent to evaluate the ability of the model to describe the squeezing conditions with reference to a real case study. The cross section of the tunnel taken for purpose of back analysis, is very suitable because the adopted support system permits the squeezing deformations to develop completely without a significant influence of the lining. Due to this, the deformations of the tunnel are connected only to the time dependent behaviour of the rock mass.

Preliminary numerical analyses performed by using the viscoplastic parameters calibrated on the laboratory tests on coal show that this set of parameters is not able to reproduce correctly the time dependent deformations of the tunnel. This is essentially due to the scale effect and to the heterogeneity of the rock mass which is composed by a large number of rock types. Therefore, a series of calibration analyses was carried out in order to find an appropriate set of constitutive parameters that can reproduce the monitored displacements.

The results of the numerical analysis are compared with the monitoring data of convergences, radial displacements and longitudinal displacements. It is possible to state that the agreement of the numerical results with the mean monitored data is excellent, notwithstanding the scattering of the monitoring data due to the heterogeneity and anisotropy of the rock mass. Therefore, it is possible to conclude that the ability of the SHELVIP model to describe the squeezing conditions is verified successfully. Also the results of stress distribution around the tunnel perimeter are satisfactory and reliable.

It is possible to conclude that the SHELVIP model can be used with confidence, in order to reproduce by numerical analysis the behaviour of tunnels under severe squeezing conditions.
8.3  Recommendations for further developments

Further developments are needed and some open questions remain to be addressed for the study of the time dependent behaviour of weak rocks in relation to tunnel excavation, in particular for the assessment of the stability conditions of both the face and the heading, the timely installation of the tunnel support, and the excavation method.

- Further tests on coal are needed in order to confirm the experimental observations, with particular attention to low confinement stress field, stress dependence and tertiary creep.

- As clearly seen in Chapter 7 most of time dependent deformations develop after the yielding of the material. The post failure time dependent behaviour is completely unknown and need to be studied by means of special tests.

- The tertiary phase of creep is to be implemented into the SHELVIP model, by introducing a viscoplastic hardening of the plastic yield surface.

- The ability of the SHELVIP constitutive model to describe the interaction between the rock mass and the support system need to be evaluated by means of numerical back analyses. Three-dimensional analyses are recommended.

- The simulation of mechanized excavation (Tunnel Boring Machine) in severe squeezing conditions by using the SHELVIP model is desirable in order to gain in the understanding of the tunnel response during face advance.

- The long term behaviour of the rock mass (few years) is still a problematic open question and need to be studied by means of laboratory tests, in situ measurements, and numerical analyses.
Appendix A

List of symbols

The derivative of a quantity $x$ over time $t$ is indicated as $\dot{x}$.

**Soil mechanics parameters:**

- $C_{ijkl}$: compliance matrix
- $\varepsilon_{ij}$: deviator strain tensor
- $E$: Young's modulus
- $E_{ijkl}$: stiffness matrix
- $G$: shear modulus
- $K$: bulk modulus
- $p$: mean stress
- $q$: deviatoric stress
- $s$: mean stress
- $s_{ij}$: deviator stress tensor
- $t$: time; deviatoric stress
- $u$: pore pressure
- $W$: work
- $\gamma$: distortional strain
- $\delta_{ij}$: Kronecker delta
- $\varepsilon_1; \varepsilon_2; \varepsilon_3$: principal strains $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$
- $\varepsilon$: strain
- $\varepsilon_{ij}$: strain tensor
- $\nu$: Poisson's ratio
- $\sigma$: stress
- $\sigma_1; \sigma_2; \sigma_3$: principal stresses $\sigma_1 > \sigma_2 > \sigma_3$
- $\sigma_{ij}$: stress tensor
- $\tau$: shear stress
Appendix A. List of symbols

Apex:

· effective value
·B Bingham value
·c creep value
·D value relative to the dashpot
·e elastic value
·ep elastoplastic value
·K Kelvin value
·I first trial value
·M Maxwell value
·N new value
·O old value
·p plastic value
·s shear yielding value
·S value relative to the slider
·t tensile yielding value
·v viscous value
·ve viscoelastic value
·vp viscoplastic value
·η viscous value
·(0) initial value
·(t) value at time \( t \)

Pedex:

·’0 initial or reference value; value for \( t \rightarrow 0 \); unitary value
·’0,1,2,3... different values
·’A,B,C... different values
·’a axial value
·’const constant value
·’e value referred to the void index
·’i initial or reference value; value of test \( i \)
·’ini value for the onset of viscoplastic behaviour
·’max maximum value
·’min minimum value
·’p volumetric value
·’q deviatoric value
·’r radial value
·’t value at time \( t \)
Appendix A. List of symbols

\begin{itemize}
  \item \( v \) \hspace{1cm} \text{vertical value}
  \item \( z \) \hspace{1cm} \text{vertical value}
  \item \( \varepsilon \) \hspace{1cm} \text{value referred to the vertical strain}
  \item \( \infty \) \hspace{1cm} \text{value for } t \to \infty
\end{itemize}

Other parameters:

\begin{itemize}
  \item \( a \) \hspace{1cm} \text{parameter of Lemaitre’s model}
  \item \( a_2 \) \hspace{1cm} \text{parameter of Yin and Graham’s model}
  \item \( A \) \hspace{1cm} \text{parameter of Singh and Mitchell’s model; parameter of}
  \hspace{1cm} \text{Tavena’s model; parameter}
  \item \( b \) \hspace{1cm} \text{parameter of Prevost’s relaxation model}
  \item \( b_p \) \hspace{1cm} \text{coefficient}
  \item \( bp \) \hspace{1cm} \text{back pressure}
  \item \( B \) \hspace{1cm} \text{parameter of Tavena’s model; Skempton’s parameter}
  \item \( c \) \hspace{1cm} \text{cohesion; constant}
  \item \( C \) \hspace{1cm} \text{constant; percent clay sizes}
  \item \( C_c \) \hspace{1cm} \text{compression index}
  \item \( C_p \) \hspace{1cm} \text{velocity of computational wave}
  \item \( C_r \) \hspace{1cm} \text{recompression index}
  \item \( C_\alpha \) \hspace{1cm} \text{secondary compression coefficient}
  \item \( D \) \hspace{1cm} \text{stiffness matrix; deviator stress}
  \item \( e \) \hspace{1cm} \text{void index}
  \item \( E_{edo} \) \hspace{1cm} \text{oedometric modulus}
  \item \( E_j \) \hspace{1cm} \text{FDM element}
  \item \( f \) \hspace{1cm} \text{function; yield function}
  \item \( f_p \) \hspace{1cm} \text{plastic yield function}
  \item \( f_{vp} \) \hspace{1cm} \text{viscoplastic yield function}
  \item \( f' \) \hspace{1cm} \text{part of the yield function which do not depend on time}
  \item \( f \) \hspace{1cm} \text{part of the yield function which depends on the effective}
  \hspace{1cm} \text{state of stress}
  \item \( f_p \) \hspace{1cm} \text{plastic yield function}
  \item \( f_{vp} \) \hspace{1cm} \text{viscoplastic yield function}
  \item \( F \) \hspace{1cm} \text{overstress function; force}
  \item \( F_i \) \hspace{1cm} \text{nodal forces}
  \item \( F_{di} \) \hspace{1cm} \text{unbalanced nodal forces}
  \item \( g \) \hspace{1cm} \text{function; potential function}
  \item \( g_p \) \hspace{1cm} \text{plastic potential function}
  \item \( g_{vp} \) \hspace{1cm} \text{viscoplastic potential function}
  \item \( g_i \) \hspace{1cm} \text{vector of volumetric accelerations}
  \item \( G_s \) \hspace{1cm} \text{specific grain weight}
\end{itemize}
Appendix A. List of symbols

\( h \) function
\( h_p \) function
\( h_{vp} \) function
\( i \) test number
\( IP \) plastic index
\( J_p^{\text{vp}} \) second invariant of deviatoric viscoplastic strains
\( k_0 \) lateral earth pressure
\( k_p \) intercept of the Drucker-Prager plastic criterion
\( l \) constitutive parameter of SHELVIP model
\( L \) loading operator of the NSFS theory
\( m \) constitutive parameter of SHELVIP model; Singh and Mitchell’s parameters for creep; constitutive parameter of Lemaitre’s model; mass
\( m' \) ratio \( C_\alpha /C_c \)
\( n \) constitutive parameter; constitutive parameter of SHELVIP model; constitutive parameter of Lemaitre’s model; porosity
\( n_1 \) parameter of Yin and Graham’s model
\( n_3 \) parameter of Yin and Graham’s model
\( \bar{q} \) normalized deviatoric stress
\( r_p \) function
\( s \) parameter of Lacerda and Houston’s model
\( S \) saturation
\( t_e \) equivalent time for the Yin and Graham’s model
\( t_f \) time to failure
\( T \) temperature
\( u_i \) nodal displacement
\( w \) water content
\( w_n \) natural water content
\( w_L \) liquid limit
\( w_P \) plastic limit
\( x \) generic variable
\( x_i \) nodal position
\( \alpha \) parameter of Singh and Mitchell’s creep model; constitutive parameter of Lemaitre’s model
\( \alpha_d \) damping constant
\( \alpha_p \) slope of the Drucker-Prager plastic criterion
\( \alpha_{vp} \) slope of the Drucker-Prager viscoplastic criterion; viscoplastic hardening level
\( \bar{\alpha} \) parameter of Singh and Mitchell’s creep model
\( \beta \) constitutive parameter of Lemaitre’s model; time dependent parameter of NSFS theory; parameter
### Appendix A. List of symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_p )</td>
<td>coefficient</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>fluidity parameter; volumetric weight; constitutive parameter of SHELVIP model</td>
</tr>
<tr>
<td>( \Delta t )</td>
<td>time step</td>
</tr>
<tr>
<td>( \Delta t_c )</td>
<td>critical time step</td>
</tr>
<tr>
<td>( \Delta x )</td>
<td>edge of square FDM mesh</td>
</tr>
<tr>
<td>( \varepsilon_{1,0} )</td>
<td>strain before start of relaxation</td>
</tr>
<tr>
<td>( \eta )</td>
<td>dynamic viscosity</td>
</tr>
<tr>
<td>( \theta )</td>
<td>angle; parameter used in the calibration of the SHELVIP model</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>part of the yield function dependent on the viscoplastic deviatoric strain; parameter of Yin and Graham’s model; hardening function</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>parameter of Yin and Graham’s model; plastic multiplier</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>viscoplastic multiplier of NSFS theory</td>
</tr>
<tr>
<td>( \nu )</td>
<td>specific volume</td>
</tr>
<tr>
<td>( \phi )</td>
<td>friction angle; angle</td>
</tr>
<tr>
<td>( \phi_r )</td>
<td>residual friction angle</td>
</tr>
<tr>
<td>( \Phi )</td>
<td>viscous nucleus</td>
</tr>
<tr>
<td>( \rho_v )</td>
<td>volumetric weight</td>
</tr>
<tr>
<td>( \sigma_c )</td>
<td>consolidation effective stress</td>
</tr>
<tr>
<td>( \sigma_{ei} )</td>
<td>uniaxial strength</td>
</tr>
<tr>
<td>( \sigma_{lt} )</td>
<td>long term strength</td>
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<td>( \sigma_{st} )</td>
<td>short term strength</td>
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<td>yielding stress</td>
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<td>( \sigma'_y )</td>
<td>stress unit</td>
</tr>
<tr>
<td>( \sigma'_{0e} )</td>
<td>parameter of Yin and Graham’s model</td>
</tr>
<tr>
<td>( \sigma'_{z,pc} )</td>
<td>effective vertical preconsolidation pressure</td>
</tr>
<tr>
<td>( \sigma'_{z,pc1} )</td>
<td>effective vertical quasi-preconsolidation or apparent preconsolidation pressure</td>
</tr>
<tr>
<td>( \tau )</td>
<td>time function; integration variable</td>
</tr>
<tr>
<td>( \chi )</td>
<td>parameter used in the calibration of the SHELVIP model</td>
</tr>
<tr>
<td>( \psi )</td>
<td>coefficient of the Yin law for creep; parameter of Yin and Graham’s model; dilatancy angle</td>
</tr>
<tr>
<td>( \Psi )</td>
<td>parameter Lacerda and Houston’s relaxation model</td>
</tr>
<tr>
<td>( \omega_p )</td>
<td>plastic dilatancy</td>
</tr>
<tr>
<td>( \omega_{vp} )</td>
<td>viscoplastic dilatancy</td>
</tr>
</tbody>
</table>
Appendix B

Derivation of the SHELVIP model

This appendix describes the derivation of the general flow rule and the analytical expression of creep for the SHELVIP constitutive model.

General flow rule

The viscoplastic strain rate tensor can be calculated by using the flow rule of Equation (6.17):

\[
\dot{\varepsilon}_{ij}^{vp} = \gamma \cdot \Phi (F) \cdot \frac{\partial g_{vp}}{\partial \sigma_{ij}}
\]  

(B.1)

By introducing the main hypothesis about the overstress function (Equation (6.18)), the viscoplastic nucleus (Equation (6.19)) and the viscoplastic potential function (Equation (6.20)):

\[
F = f_{vp}
\]  

(B.2)

\[
\Phi (F) = \langle F \rangle^n
\]  

(B.3)

\[
g_{vp} = q - \omega_{vp} \cdot p
\]  

(B.4)

Equation (B.1) can be written as:

\[
\dot{\varepsilon}_{ij}^{vp} = \gamma \cdot \langle f_{vp} \rangle^n \cdot \left( \frac{\partial q}{\partial \sigma_{ij}} - \omega_{vp} \cdot \frac{\partial p}{\partial \sigma_{ij}} \right)
\]  

(B.5)

The derivative of \( q \) with respect to the state of stress \( \sigma_{ij} \) can be expressed as:
\[
\frac{\partial q}{\partial \sigma_{ij}} = \frac{\partial q}{\partial s_{kl}} \cdot \frac{\partial s_{kl}}{\partial \sigma_{ij}} \tag{B.6}
\]

The deviatoric stress \( q \) is expressed as:

\[
q = \sqrt{\frac{3}{2} s_{kl} \cdot s_{kl}} \tag{B.7}
\]

Therefore the first term of Equation (B.6) can be calculated as:

\[
\frac{\partial q}{\partial s_{kl}} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{s_{kl} + \frac{3}{2} s_{rl}}{\sqrt{\frac{3}{2} s_{rl} \cdot s_{rl}}} = \frac{3}{2} \cdot \frac{s_{kl}}{q} \tag{B.8}
\]

The deviatoric state of stress \( s_{kl} \) can be expressed as:

\[
s_{kl} = \sigma_{kl} - \frac{1}{3} \cdot \sigma_{mm} \cdot \delta_{kl} = \sigma_{ij} \cdot \delta_{ik} \cdot \delta_{jl} - \frac{1}{3} \cdot \sigma_{ij} \cdot \delta_{ij} \cdot \delta_{kl} \tag{B.9}
\]

The second term of Equation (B.6) can be calculated as:

\[
\frac{\partial s_{kl}}{\partial \sigma_{ij}} = \delta_{ik} \cdot \delta_{jl} - \frac{1}{3} \cdot \delta_{ij} \cdot \delta_{kl} \tag{B.10}
\]

By substituting Equations (B.8) and (B.10) into Equation (B.6) it is possible to obtain:

\[
\frac{\partial q}{\partial \sigma_{ij}} = \frac{3}{2} \cdot \frac{s_{kl}}{q} \cdot \left( \delta_{ik} \cdot \delta_{jl} - \frac{1}{3} \cdot \delta_{ij} \cdot \delta_{kl} \right) = \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{2} \cdot \frac{\delta_{ij}}{q} \cdot (s_{kl} \cdot \delta_{kl}) \tag{B.11}
\]

By remembering that the trace of \( s \) is equal to zero, \( \text{tr}(s) = s_{kl} \cdot \delta_{kl} = 0 \), it is possible to obtain:

\[
\frac{\partial q}{\partial \sigma_{ij}} = \frac{3}{2} \cdot \frac{s_{ij}}{q} \tag{B.12}
\]

The derivative of \( p \) with respect to \( \sigma_{ij} \) can be easily obtained upon the definition of the volumetric stress \( p \):

\[
p = \frac{1}{3} \cdot \sigma_{ij} \cdot \delta_{ij} \tag{B.13}
\]

Therefore it is possible to write:

\[
\frac{\partial p}{\partial \sigma_{ij}} = \frac{1}{3} \cdot \delta_{ij} \tag{B.14}
\]
By substituting Equation (B.12) and Equation (B.14) into Equation (B.5) it is possible to obtain:

\[
\dot{\varepsilon}_{ij}^{vp} = \gamma \cdot f_{vp}^{n} \cdot \left( \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right)
\]  

(B.15)

**Analytical expression for creep**

A constant state of stress \( \sigma_{ij} \), below the plastic yield surface \( f_{p}(\sigma_{ij}) < 0 \) and above the viscoplastic yield surface \( f_{vp}(\sigma_{ij}) > 0 \), is considered. Therefore the plastic deformations do not develop and the behaviour of the material is only elastic-viscoplastic.

It is necessary to evaluate the expression of \( f_{vp} \). This can be done by integrating the differential viscoplastic hardening law of Equation (6.23):

\[
\dot{\alpha}_{vp} = \frac{l}{m \cdot n} \cdot \frac{f_{vp}}{p + \frac{k_{p}}{\alpha_{p}}} \cdot \left( \frac{f_{vp}}{q} \right)^{m-n}
\]  

(B.16)

If both the terms of the Equation (B.16) are multiplied by the term \( p + k_{p}/\alpha_{p} \), that is surely different from zero, it is possible to obtain:

\[
\dot{\alpha}_{vp} \cdot \left( p + \frac{k_{p}}{\alpha_{p}} \right) = \frac{l}{m \cdot n} \cdot \dot{f}_{vp} \cdot \left( \frac{f_{vp}}{q} \right)^{n-m}
\]  

(B.17)

At this point it is important to remember the definition of \( f_{vp} \) of Equation (6.12):

\[
f_{vp} = q - \alpha_{vp} \cdot \left( p + \frac{k_{p}}{\alpha_{p}} \right)
\]  

(B.18)

Given that \( q \) and \( p \) are constant, because the state of stress \( \sigma_{ij} \) is constant with time, the time derivative of Equation (B.18) leads to:

\[
\dot{f}_{vp} = -\dot{\alpha}_{vp} \cdot \left( p + \frac{k_{p}}{\alpha_{p}} \right)
\]  

(B.19)

If the Equation (B.19) is introduced into Equation (B.17) one obtains:

\[
-\dot{f}_{vp} = \frac{l}{m \cdot n} \cdot \dot{f}_{vp} \cdot \left( \frac{f_{vp}}{q} \right)^{m-n}
\]  

(B.20)

By subdividing the time derivative Equation (B.20) can be written as:

\[
- f_{vp}^{m-n-1} \cdot df_{vp} = \frac{l}{m \cdot n} \cdot q^{-m-n} \cdot dt
\]  

(B.21)
which can be integrated:

\[ f_{vp}^{-m} = l \cdot q_{vp}^{-m} \cdot t + c_1 \quad (B.22) \]

where \( c_1 \) is a integration constant that can be determined by assuming for the initial time:

\[ f_{vp} = f_{vp,0} \quad : \quad t = 0 \quad (B.23) \]

where:

\[ f_{vp,0} = q - \alpha_{vp,0} \cdot \left( p + \frac{k_p}{\alpha_p} \right) \quad (B.24) \]

where \( \alpha_{vp,0} \) is the initial hardening level.

Rewriting Equation (B.22) for the initial conditions of Equation (B.23), it is possible to obtain a value of the constant \( c_1 \):

\[ c_1 = f_{vp,0}^{-m} \quad (B.25) \]

By replacing the value of \( c_1 \) into Equation (B.23) it is possible to obtain:

\[ f_{vp}^{-m} = l \cdot q_{vp}^{-m} \cdot t + f_{vp,0}^{-m} \quad (B.26) \]

that can be solved for \( f_{vp} \):

\[ f_{vp} = q \cdot \left[l \cdot t + \left( \frac{q_{vp,0}}{f_{vp,0}} \right)^{m-n} \right]^{-\frac{1}{m-n}} \quad (B.27) \]

By substituting Equation (B.27) into Equation (B.15) it is possible to obtain the final relationship for the viscoplastic strain rate:

\[ \dot{\varepsilon}_{ij}^{vp} = \gamma \cdot q^n \cdot \left[l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^{m-n} \right]^{-\frac{1}{m-n}} \cdot \left( \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right) \quad (B.28) \]

Now it is necessary to integrate Equation (B.28) over time. This can be done in a quite easy way because the stress terms \( q, p, s_{ij} \) are constant. This can be done by distinguishing the case \( m \neq 1 \) and the case \( m = 1 \).

In the case \( m \neq 1 \) the integration leads to:

\[ \varepsilon_{ij}^{vp} = \frac{\gamma}{l} \cdot \frac{m}{m-1} \cdot q^n \cdot \left[l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^{m-n} \right]^{\frac{m-1}{m}} \cdot \left( \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right) + c_2 \quad (B.29) \]
where $c_2$ is a integration constant that can be determined by assuming for the initial time:

$$\varepsilon^\text{vp}_{ij} = 0 : \quad t = 0 \quad (B.30)$$

This gives:

$$c_2 = -\frac{\gamma}{l} \cdot \frac{m}{m-1} \cdot q^n \cdot \left( \frac{q}{f_{vp,0}} \right)^{n \cdot (m-1)} \cdot \left[ \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right] \quad (B.31)$$

By substituting the expression of $c_2$ into Equation (B.29) it is possible to obtain the expression for the viscoplastic strain:

$$\varepsilon^\text{vp}_{ij} = \frac{\gamma}{l} \cdot \frac{m}{m-1} \cdot q^n \cdot \left\{ \left[ l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^{m-1} \right] - \left( \frac{q}{f_{vp,0}} \right)^{n \cdot (m-1)} \right\} \cdot \left[ \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right] \quad (B.32)$$

In the case of $m = 1$, Equation (B.28) is reduced to:

$$\dot{\varepsilon}^\text{vp}_{ij} = \frac{\gamma}{l} \cdot q^n \cdot \frac{1}{l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^n} \cdot \left[ \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right] \quad (B.33)$$

its integration over time leads to:

$$\varepsilon^\text{vp}_{ij} = \frac{\gamma}{l} \cdot q^n \cdot \ln \left[ l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^n \right] \cdot \left[ \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right] + c_3 \quad (B.34)$$

where $c_3$ is a integration constant that can be determined by assuming for the initial time:

$$\varepsilon^\text{vp}_{ij} = 0 : \quad t = 0 \quad (B.35)$$

This gives:

$$c_3 = -\frac{\gamma}{l} \cdot q^n \cdot \ln \left( \frac{q}{f_{vp,0}} \right)^n \cdot \left[ \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right] \quad (B.36)$$

By substituting this expression of $c_3$ into Equation (B.34) it is possible to obtain the expression for the viscoplastic strain:
Appendix B. Derivation of the SHELVIP model

\[ \varepsilon_{ij}^{vp} = \frac{\gamma}{l} \cdot q^n \cdot \left\{ \ln \left[ l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^m \right] - \ln \left( \frac{q}{f_{vp,0}} \right)^n \right\} \cdot \left( \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right) \]  

or in a better form:

\[ \varepsilon_{ij}^{vp} = \frac{\gamma}{l} \cdot q^n \cdot \ln \left[ 1 + l \cdot t \cdot \left( \frac{f_{vp,0}}{q} \right)^n \right] \cdot \left( \frac{3}{2} \cdot \frac{s_{ij}}{q} - \frac{1}{3} \cdot \omega_{vp} \cdot \delta_{ij} \right) \]  

(B.37)

For a triaxial creep test the state of stress is characterized only by the axial stress \( \sigma_a \) and by the radial stress \( \sigma_r \). Therefore the deviatoric stress \( q \) and the volumetric stress \( p \) can be calculated as:

\[ q = \sigma_a - \sigma_r \]  

(B.39)

\[ p = \frac{1}{3} (\sigma_a + 2 \cdot \sigma_r) \]  

(B.40)

The deviatoric states of stress can be evaluated as:

\[ s_a = \sigma_a - \frac{1}{3} (\sigma_a + 2 \cdot \sigma_r) = \frac{2}{3} \cdot (\sigma_a - \sigma_r) = \frac{2}{3} \cdot q \]  

(B.41)

\[ s_r = \sigma_r - \frac{1}{3} (\sigma_a + 2 \cdot \sigma_r) = -\frac{1}{3} \cdot (\sigma_a - \sigma_r) = -\frac{1}{3} \cdot q \]  

(B.42)

The axial viscoplastic strain rate \( \dot{\varepsilon}_{a}^{vp} \) and the radial viscoplastic strain rate \( \dot{\varepsilon}_{r}^{vp} \) can be expressed as:

\[ \dot{\varepsilon}_{a}^{vp} = \gamma \cdot q^n \cdot \left[ l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^m \right]^{-\frac{1}{m}} \cdot \left( 1 - \frac{1}{3} \cdot \omega_{vp} \right) \]  

(B.43)

\[ \dot{\varepsilon}_{r}^{vp} = -\gamma \cdot q^n \cdot \left[ l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^m \right]^{-\frac{1}{m}} \cdot \left( \frac{1}{2} + \frac{1}{3} \cdot \omega_{vp} \right) \]  

(B.44)

The axial viscoplastic strain \( \varepsilon_{a}^{vp} \) and the radial viscoplastic strain \( \varepsilon_{r}^{vp} \) can be calculated as follows for \( m \neq 1 \):

\[ \varepsilon_{a}^{vp} = \frac{\gamma}{l} \cdot \frac{m}{m-1} \cdot q^n \cdot \left\{ l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^m \right\}^{-\frac{m-1}{m}} \cdot \left( \frac{q}{f_{vp,0}} \right)^{n \cdot (m-1)} \]  

\[ \cdot \left( 1 - \frac{1}{3} \cdot \omega_{vp} \right) \]  

(B.45)
\[ \varepsilon_{vp}^{r} = -\frac{\gamma}{l} \cdot \frac{m}{m-1} \cdot q^n \cdot \left\{ \left[ l \cdot t + \left( \frac{q}{f_{vp,0}} \right)^{m-n} \right]^{\frac{m-1}{m}} - \left( \frac{q}{f_{vp,0}} \right)^{n \cdot (m-1)} \right\} \cdot \left( \frac{1}{2} + \frac{1}{3} \cdot \omega_{vp} \right) \]  

In the case \( m = 1 \) it is possible to obtain:

\[ \varepsilon_{vp}^{a} = \frac{\gamma}{l} \cdot q^n \cdot \ln \left[ 1 + l \cdot t \cdot \left( \frac{f_{vp,0}}{q} \right)^n \right] \cdot \left( 1 - \frac{1}{3} \cdot \omega_{vp} \right) \quad (B.47) \]

\[ \varepsilon_{vp}^{r} = -\frac{\gamma}{l} \cdot q^n \cdot \ln \left[ 1 + l \cdot t \cdot \left( \frac{f_{vp,0}}{q} \right)^n \right] \cdot \left( \frac{1}{2} + \frac{1}{3} \cdot \omega_{vp} \right) \quad (B.48) \]
Bibliography


Bibliography


